The $\mathbb{Z}_2$ staggered vertex model and its applications

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Abstract

New solvable vertex models can be easily obtained by staggering the spectral parameter in already known ones. This simple construction reveals some surprises: for appropriate values of the staggering, highly non-trivial continuum limits can be obtained. The simplest case of staggering with period 2 (the $\mathbb{Z}_2$ case) for the six-vertex model was shown to be related, in one regime of the spectral parameter, to the critical antiferromagnetic Potts model on the square lattice, and has a non-compact continuum limit. Here we study the other regime: in the very anisotropic limit, it can be viewed as a zig–zag spin chain with spin anisotropy. From the Bethe–Ansatz solution, we obtain the central charge $c = 2$, the conformal spectrum and the continuum partition function, corresponding to one free boson and two Majorana fermions. Finally, moving in more physical territory, we obtain a massive integrable deformation of the model on the lattice. Interestingly, its scattering theory is a massive version of the one for the flow between minimal models. The corresponding field theory is argued to be a complex version of the $C_2^{(2)}$ Toda theory.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is a simple consequence of the quantum inverse scattering [1] formalism (going back to Baxter's 'Z invariance' [2]) that new integrable vertex models can be obtained from basic ones by allowing for some staggering of the spectral parameters. If the basic $\check{R}$-matrix is associated with a single crossing, one can in this way build 'block' $\check{R}$-matrices, using p^2 crossings, with $p = 2, 3, \ldots$, and staggering the spectral parameters (see figure 1).
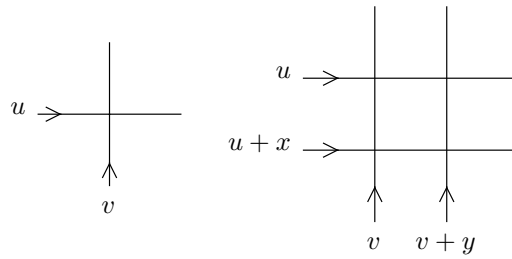
Figure 1. The $\check{R}$-matrices for a basic vertex model (left), and for a staggered vertex model with $p = 2$ (right).

While constructing the model and writing down the Bethe–Ansatz equations (BAE) is straightforward, the physics of the models thus obtained presents interesting subtleties. A striking example, in relation with the antiferromagnetic Potts model [3], was discussed in detail in [4, 5]. It was found then that the case $n = 2$ for the six-vertex model has a non-compact continuum limit [5] in a certain regime of the spectral parameter (see below for a more accurate definition), and may be related to the complex sine-Gordon (SG) model. While major difficulties remain in this case, the other regime of the spectral parameter turns out to be also of interest, and somewhat more tractable. Its study is the main goal of this paper.

1.1. Vertex models, spin chains and field theories

The correspondence between integrable spin chains with $SU(2)$ or $SU(2)_q$ symmetry and quantum field theories has been investigated in great detail already. In the antiferromagnetic regime, it is well known that a chain of spin s corresponds to a level-2s Weiss–Zumino–Witten (WZW) model in the $SU(2)$ case, and to a deformation of this theory in the Cartan direction for $SU(2)_q$ [6]. Cursory examination of the literature would suggest that nothing much remains to be done in this area.

From the R-matrix point of view, models with higher values of the spin s are obtained by projecting tensor products of ($2s$) fundamental representations of $U_q(\mathrm{sl}_2)$ onto higher spin. This construction has a close parallel in conformal field theory (CFT), where higher level ($2s$) representations of the current algebra are obtained by combining ($2s$) level one representations. There are several reasons why it would be interesting to build integrable models which are *not* projected onto irreducible $U_q(\mathrm{sl}_2)$ components. In the case $s = 1$ for instance, this would correspond to models of pairs of spin-$\frac{1}{2}$ variables. This can be reinterpreted more physically in terms of ladders, or in terms of electron physics, pairs of wires or channels. The latter case is of the highest importance. For instance, the two-channel Kondo model [7] or the two-channel interacting resonant level models [8, 9] are usually solved by going to an even–odd basis, which effectively amounts to solving the problem in the level $k = 2$ sector. Many physical questions are however related to the mixture of the even and odd degrees of freedom—e.g. because it corresponds to transport of electrons between wires. The search for integrable cases where this mixture could be studied is a priority.

To have a better idea of what to expect, it is useful to turn to the CFT point of view. Imagine starting with two $SU(2)$ Kac–Moody algebras at level 1, represented by the currents j_i^μ, $i = 1, 2$ and $\mu = 1, 2, 3$. The sums

$$J^\mu = j_1^\mu + j_2^\mu \tag{1.1}$$

are well known to provide then a Kac–Moody algebra at level 2. Of course, each level 1 corresponds to the central charge $c = 1$, while level 2 has the central charge $c = 3/2$. The point is that in taking two copies of level 1, an Ising model CFT factors out, according to the well-known decomposition

$$SU(2)_1 \times SU(2)_1 = SU_2(2) + \text{Ising}. \tag{1.2}$$

This can be illustrated quickly using bosonization. Introduce two chiral bosons ϕ_1, ϕ_2 with propagators

$$\langle \phi_i(z)\phi_i(w) \rangle = -\frac{1}{4\pi} \ln(z - w). \tag{1.3}$$

The two level-1 current algebras are obtained through

$$j_i^{\pm} \propto \exp(\pm i\sqrt{8\pi}\phi_i), \qquad j_i^3 \propto \partial\phi_i. \tag{1.4}$$

It is convenient to introduce now symmetric and antisymmetric combinations of the bosons:

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2), \qquad \phi = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2). \tag{1.5}$$

So we have

$$J^{\pm} \propto \cos\sqrt{4\pi}\phi \, \exp(\pm i\sqrt{4\pi}\Phi), \qquad J^3 \propto \partial\Phi. \tag{1.6}$$

The field $\cos(\sqrt{4\pi}\Phi)$ is a Majorana fermion [10]. The field $i \sin(\sqrt{4\pi}\phi)$ is another one, which is orthogonal to the currents J^{μ}, and is discarded in the construction of $SU(2)_2$. Corresponding to this splitting, the sum $H = H_1 + H_2$ of the two one-boson Hamiltonians decomposes as $H = H_{SU(2)_2} + H_{\text{Ising}}$ where $H_{SU(2)_2} \propto \sum_{\mu} : J^{\mu}J^{\mu} :$, and J^{μ} are the currents at level 2.

Quantum deformations of $SU(2)_2$ are obtained by adding to the Hamiltonian $H_{SU(2)_2}$ a Cartan deformation proportional to J^3J^3. We can as well deform the full Hamiltonian, obtaining in this way a theory made of two Majorana fermions and one boson with anisotropy-dependent radius. This should be the continuum limit of the models we are after. We will show in the following that these models are obtained by the general staggering construction, with $n = 2$, and an appropriate choice of the spectral parameters.

Interestingly, it can be shown [11] that the staggered models correspond algebraically to solutions of the Yang–Baxter equations based on 'bigger' irreducible representations of the quantum affine algebra $U_q(\widehat{sl}_2)$. This occurs ultimately because finite-dimensional irreducible representations of quantum affine algebras are isomorphic to the products of evaluation representations, which are themselves 'decorations' (with the spectral parameter) of the usual spin-s representations of $U_q(sl_2)$ [12].

1.2. Massive deformation

Another important application of the staggered model is that it can be used to produce a lattice discretization of a *massive* QFT. This is done, following [13], by introducing into the staggered model an additional (purely imaginary) staggering of the spectral parameters. Using this approach, we obtain a scattering theory involving two types of massive particles, where the scattering between particles of the same type (resp. different types) is given by the sine-Gordon S-matrix (resp. the sine-Gordon S-matrix with an imaginary shift in the rapidity). It turns out that such a scattering theory (but with massless particles) arose before [14] in the context of minimal models of CFT perturbed by the Φ_{13} operator. Consider the action

$$A = A_{\text{min}} + \lambda \int d^2x \, \Phi_{13}(x), \tag{1.7}$$

where $A_{\min}$ is the action of a minimal model of CFT. The problem of finding the thermodynamic Bethe–Ansatz (TBA) equations for the renormalization-group flow of theory (1.7) was first studied by Zamolodchikov in [15, 16]. The results depend crucially on the sign of the coupling λ. For $\lambda < 0$, model (1.7) becomes massive. It was shown in [15] that the corresponding S-matrix is the simple RSOS S-matrix and that the TBA diagram is of the A_n type, with a massive particle at one end of the diagram. For $\lambda > 0$, model (1.7) describes the massless flow between two consecutive minimal models. In [16], a TBA diagram was proposed, without resorting to an S-matrix: this diagram is also of the A_n type, but with mass terms $e^{\pm\theta}$ at the two ends of the diagram. It was found later, in [14], that the corresponding scattering theory consists of massless left/right particles, interacting through the SG and shifted-SG S-matrices.

In summary, we construct here a non-critical lattice model whose continuum excitations are described by a massive version of the S-matrix for the flow between minimal models of CFT. We also propose an effective QFT for this model, with one boson and two Majorana fermions which interact with each other.

We note that the results of this section are somewhat less complete and rigorous than those of the other sections. On the other hand, they form a nice continuation, and provide potential physical applications of our model. We note that the S-matrix we obtain is a new object, which bears intriguing potential relationship with as yet unsolved models, such as the Lund Regge field theory.

1.3. Outline

In section 2, we expose in more detail the construction of the model and its various lattice formulations, as well as the relation with spin chains. In section 3, we present the Bethe–Ansatz solution, and obtain the critical exponents, through the study of low-energy excitations. In section 4, we discuss the associated CFT, and exhibit the full operator content through the study of torus partition functions. Finally, in section 5, we study the integrable massive deformation, and derive its scattering theory, TBA equations and ground-state energy scaling function. This allows us to propose an interacting effective QFT. Some important but quite long calculations are done in the appendices.

2. Solvable models based on the Temperley–Lieb algebra

In this section, we explain the construction of the $\mathbb{Z}_2$ staggered model from the Temperley–Lieb algebra. It appears as a solution of the star-triangle equations for the Potts model in the antiferromagnetic regime [3, 17]. We recall the relation of the TL algebra to Potts and RSOS models, and give some comments on the anyonic fusion interpretation of RSOS models. Finally, we present the quantum-chain Hamiltonian related to the $\mathbb{Z}_2$ staggered model.

2.1. Temperley–Lieb algebra and Yang–Baxter equations

We consider a square lattice of width $2N$ sites. Let us recall the algebraic relations defining the Temperley–Lieb (TL) algebra [18] with generators e_j ($j = 1, \ldots, 2N - 1$) and loop weight n:

$$e_j^2 = n\, e_j$$
$$e_j e_{j\pm1} e_j = e_j \tag{2.1}$$
$$e_j e_{j'} = e_{j'} e_j \qquad \text{if } |j - j'| > 1.$$

In this paper, we are interested in the regime $0 \leqslant n \leqslant 2$, so we set the notation:

$$n = q + q^{-1}, \qquad q = e^{i\gamma}, \qquad 0 \leqslant \gamma \leqslant \frac{\pi}{2}. \tag{2.2}$$
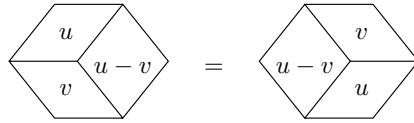
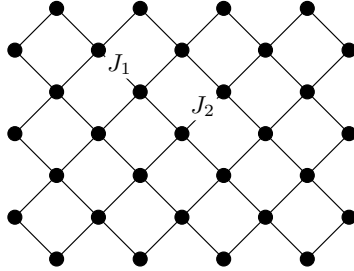**Figure 2.** The Yang–Baxter equations (2.5).

Figure 3. The Potts model on the square lattice with anisotropic couplings J_1, J_2. The dots represent Potts spins S_j.

Using the TL algebra, one constructs the central object of lattice integrability, which is the $\check{R}$-matrix acting between sites j and $j + 1$:

$$\check{R}_{j,j+1}(u) \equiv \sin(\gamma - u)\, 1 + \sin u\, e_j. \tag{2.3}$$

This is depicted as

$$\tag{2.4}$$

The $\check{R}$-matrix then satisfies the Yang–Baxter equations (see figure 2) [19]:

$$\check{R}_{j,j+1}(u)\check{R}_{j+1,j+2}(u - v)\check{R}_{j,j+1}(v)$$
$$= \check{R}_{j+1,j+2}(v)\check{R}_{j,j+1}(u - v)\check{R}_{j+1,j+2}(u). \tag{2.5}$$

The inhomogeneous partition function on a square lattice is obtained by placing a plaquette (2.4) at coordinates (x, y), with parameter $u = u_y - v_x$, where the u_y (resp. v_x) are the horizontal (resp. vertical) spectral parameters on the lattice.

2.2. Relation to the Potts model

The Q-state Potts model [20] is a model of classical spins with nearest-neighbour interactions. Each spin S_j can take Q values, and sits on a vertex of the square lattice. The partition function is given by

$$Z_{\text{Potts}} = \sum_{\{S_j=1,\dots,Q\}} \prod_{i \diagdown j} \exp(J_1 \delta_{S_i,S_j}) \prod_{i \diagup j} \exp(J_2 \delta_{S_i,S_j}), \tag{2.6}$$

where each product runs on edges with the corresponding orientation, J_1 and J_2 are the couplings and δ is the Kronecker symbol (see figure 3).

Since the work of Baxter [3, 19], it has been known that the Potts model is equivalent to an integrable TL loop model (with loop weight $n = \sqrt{Q}$) in two cases, described in terms of the parameters $x_r = (e^{J_r} - 1)/\sqrt{Q}$.

- The self-dual line $x_1 x_2 = 1$. It is equivalent to a TL model with a homogeneous spectral parameter u, given by $x_1 = \sin u / \sin(\gamma - u)$. In the regime $x_1, x_2 > 0$, it describes the paramagnetic/ferromagnetic transition of the Potts model. In the regime $x_1, x_2 < 0$, it describes the Berker–Kadanoff critical phase.
- The 'staggered critical line' $(2 + x_1\sqrt{Q})(2 + x_2\sqrt{Q}) = 4 - Q$. It is equivalent to a TL model with staggered spectral parameters $\{v_x\} = \{0, \frac{\pi}{2}, 0, \frac{\pi}{2}, \cdots\}$ and $\{u_y\} = \{u, u + \frac{\pi}{2}, u, u + \frac{\pi}{2}, \cdots\}$ [3]. There are again two regimes. If $\gamma < u < \pi/2$, then both couplings are antiferromagnetic ($x_1, x_2 < 0$) and the staggered critical line describes the paramagnetic/antiferromagnetic transition [3–5]. In the other regime, $0 < u < \gamma$, the couplings are such that $x_1 x_2 < 0$: the corresponding Potts model is thus 'totally anisotropic'. The purpose of this paper is to study the continuum limit of this last regime, through the Bethe–Ansatz.

2.3. RSOS models at roots of unity, anyonic fusion

We have seen above that the TL algebra can be realized by Potts spins for the special values of the loop weight $n = \sqrt{Q}$, where Q is a positive integer. In this paragraph, we recall the results of [21] on the RSOS models, related to TL with loop weight

$$n = 2 \cos \frac{\pi}{m + 1},$$

where $m \in \{3, 4, \ldots\}$. The TL algebraic relations (2.1) can be realized on a vector space with each basis vector labelled by height variables $|h_1 \ldots h_{2N}\rangle$, subject to the local constraints:

$$h_j \in \{1, \ldots, m\}, \qquad |h_j - h_{j+1}| = 1. \tag{2.7}$$

The TL generators are given by their action on this basis:

$$e_j |h_1 \ldots h_{2N}\rangle = \delta_{h_{j-1} h_{j+1}} \sum_{|h' - h_{j+1}| = 1} \frac{\sqrt{S_{h_j} S_{h'}}}{S_{h_{j+1}}} |h_1 \ldots h' \ldots h_{2N}\rangle, \tag{2.8}$$

where $S_h \equiv \sin \frac{\pi h}{m+1}$.

In a recent paper [22], it was noted that the RSOS representation (2.7)–(2.8) of the TL algebra could be re-interpreted in terms of anyons for the $\mathbb{Z}_k$-parafermionic trial wavefunctions of Read–Rezayi [23]. This is due to the equivalence between the fusion rules of primary operators in the $\mathbb{Z}_k$-parafermionic CFT [24] and those of $U_q(sl_2)$, for $q = e^{\frac{i\pi}{k+2}}$. In [22] a 'topological symmetry' Y was introduced which is present in any Hamiltonian generated by the e_j (2.8). We identify this symmetry as the total $U_q(sl_2)$ spin, which corresponds to the definition of Y in terms of F-matrices in [22], and is known to commute with the TL algebra (for $n = q + q^{-1}$), even for generic values of q.

2.4. Staggered model

Returning to the generic construction of section 2.1, we choose a staggering of period $p = 2$ for both sets of spectral parameters (see figure 4). This amounts to building a homogeneous partition function with the block $\check{\mathcal{R}}$-matrix:

$$\check{\mathcal{R}}_{j, j+1}(u) = \check{R}_{2j+1}\left(u - \frac{\pi}{2}\right) \check{R}_{2j}(u) \check{R}_{2j+2}(u) \check{R}_{2j+1}\left(u + \frac{\pi}{2}\right), \tag{2.9}$$
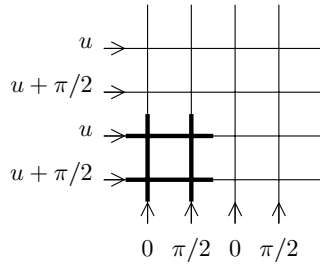
Figure 4. Spectral parameters for the staggered $\mathbb{Z}_2$ model. The thick lines represent the $\check{\mathcal{R}}$-matrix.

where $j = 1, \ldots, N$. Using definition (2.3) of $\check{R}_j(u)$ and the algebraic relations (2.1), we get

$$\check{\mathcal{R}}_{j,j+1}(u) = -\tfrac{1}{4} \sin^2(2\gamma - 2u)\, \mathbf{1}$$

$$- \tfrac{1}{2} \sin u \sin(2\gamma - 2u)[\cos(\gamma - u)(e_{2j-1} + e_{2j+1}) + 2 \cos \gamma \cos u\, e_{2j}]$$

$$+ \tfrac{1}{4} \sin 2u \sin(2\gamma - 2u)\, (e_{2j-1}e_{2j} + e_{2j}e_{2j-1} + e_{2j}e_{2j+1} + e_{2j+1}e_{2j})$$

$$+ \sin^2 u \cos u\, [\cos(\gamma - u)\, (e_{2j-1}e_{2j+1}e_{2j} + e_{2j}e_{2j-1}e_{2j+1})$$

$$- \cos u\, e_{2j}e_{2j-1}e_{2j+1}e_{2j}]. \tag{2.10}$$

The particular choice of the shift $\frac{\pi}{2}$ in the spectral parameters ensures an additional $\mathbb{Z}/2\mathbb{Z}$ symmetry: from the YBE, one can show that the two-row transfer matrix commutes with the $\mathbb{Z}_2$ charge C defined as

$$C \equiv \prod_{j=1}^{N} \frac{\check{R}_{2j-1,2j}\left(\frac{\pi}{2}\right)}{\cos \gamma}, \qquad C^2 = 1. \tag{2.11}$$

2.5. Relation to quantum chains in the very anisotropic limit

An interesting aspect of Yang–Baxter integrable statistical models is that the transfer matrix generally possesses a *very anisotropic limit*, where its derivatives with respect to u are local, one-dimensional Hamiltonians. Expression (2.10) implies that $\check{\mathcal{R}}_{j,j+1}(0) = -\tfrac{1}{4} \sin^2 2\gamma\, \mathbf{1}$, and so the two-row transfer matrix $\mathcal{T}(u)$ reduces to a cyclic translation of two sites to the right in the limit $u \to 0$. The first-order Hamiltonian is

$$H \equiv -\frac{1}{2} \sin 2\gamma\, \left.\frac{\mathrm{d} \log \mathcal{T}(u)}{\mathrm{d}u}\right|_{u=0}$$

$$= -\frac{1}{2} \sin 2\gamma\, \sum_{j=1}^{N} \check{\mathcal{R}}_{j,j+1}(0)^{-1} \frac{\mathrm{d}\check{\mathcal{R}}_{j,j+1}}{\mathrm{d}u}(0). \tag{2.12}$$

The expression of H in terms of TL generators is

$$H = 2N \cos 2\gamma\, \mathbf{1} + \sum_{j=1}^{2N} (-2 \cos \gamma\, e_j + e_j e_{j+1} + e_{j+1} e_j). \tag{2.13}$$
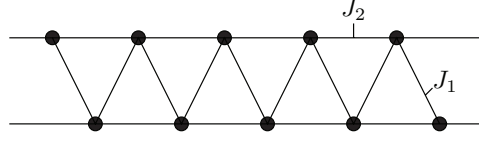
Figure 5. Zig–zag spin chain with J_1 and J_2 interactions.

This is an integrable point for the quadratic TL Hamiltonian studied in [25]:

$$H_{\mathrm{qu}}(K_1, K_2) = K_1 \sum_{j=1}^{2N} e_j + K_2 \sum_{j=1}^{2N} (e_j e_{j+1} + e_{j+1} e_j) \qquad (2.14)$$

$$H = 2N \cos 2\gamma\, 1 + H_{\mathrm{qu}}(-2 \cos \gamma, 1). \qquad (2.15)$$

Note that, besides H, $H_{\mathrm{qu}}(K_1, K_2)$ contains another remarkable point $H_{q-\mathrm{MG}} = H_{\mathrm{qu}}(-\cos \gamma, 1)$ [26], which is a q-deformation of the Majumdar–Ghosh chain [27, 28]. Like the isotropic ($q = 1$) Majumdar–Ghosh chain, $H_{q-\mathrm{MG}}$ exhibits spontaneous dimerization and a finite energy gap for q close enough to 1 [25].

For a generic value of the loop weight n, the algebraic relations (2.1) are realized by a spin-$\frac{1}{2}$ chain with $\mathrm{U}_q(\mathrm{sl}_2)$ symmetry, where $n = q + q^{-1}$. If we denote $\sigma_1, \ldots, \sigma_{2N}$ the Pauli matrices on the chain, then the projector of spins σ_j, σ_{j+1} onto the $\mathrm{U}_q(\mathrm{sl}_2)$ singlet reads, up to a multiplicative constant,

$$e_j = -\left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right) + \tfrac{1}{2}\left(1 - \sigma_j^z \sigma_{j+1}^z\right) \mathrm{e}^{\mathrm{i}\gamma \sigma_{j+1}^z}, \qquad (2.16)$$

and the above operators satisfy the TL algebraic relations. Using this representation, the first summand in (2.14) would simply give the *XXZ* spin chain. The quadratic terms in e_j, e_{j+1} lead to a different spin model

$$H_{\mathrm{qu}}(K_1, K_2) = \sum_{j=1}^{2N} \left\{ \frac{J_2}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + J_1^{xy} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right) \right.$$
$$\left. + J_1^z \sigma_j^z \sigma_{j+1}^z + \mathrm{i} J_3 \left(\sigma_{j-1}^z - \sigma_{j+2}^z\right)\left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right) \right\}, \qquad (2.17)$$

where

$$J_1^{xy} = -(K_1 + 2 \cos \gamma\, K_2), \qquad J_1^z = -\left(\tfrac{1}{2} \cos \gamma\, K_1 + K_2\right),$$
$$J_2 = K_2, \qquad J_3 = K_2 \sin \gamma.$$

The J_2 term in equation (2.17) represents two *XXX* spin chains, living on the even and odd sites respectively. The J_1 terms correspond to an *XXZ* interaction with a 'zig–zag' shape (see figure 5). The J_3 term is an anti-Hermitian three-spin interaction, with no obvious physical interpretation.

For clarity, we give the explicit expression of our integrable Hamiltonian H:

$$H = \sum_{j=1}^{2N} \left\{ \frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} - \sin^2 \gamma\, \sigma_j^z \sigma_{j+1}^z \right.$$
$$\left. + \mathrm{i} \sin \gamma \left(\sigma_{j-1}^z - \sigma_{j+2}^z\right)\left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right) \right\}. \qquad (2.18)$$

To end this section, let us discuss the consequences of a naive bosonization [29] argument for this model. We can discard the irrelevant $\mathrm{i} \sin \gamma$ term, and we end up with two free

bosons ϕ_1, ϕ_2 with compactification radii $R_1 = R_2 = R$ independent from γ, coupled by the quadratic term $\partial\phi_1\partial\phi_2$. Since this term is symmetric in the exchange of the bosons ($\phi_1 \leftrightarrow \phi_2$), the symmetric and antisymmetric combinations $\Phi = (\phi_1 + \phi_2)/\sqrt{2}$, $\phi = (\phi_1 - \phi_2)/\sqrt{2}$ are decoupled free bosons, with compactification radii $R_\pm = R \pm \delta R$, where δR depends on γ. Thus, we obtain two decoupled free bosons Φ, ϕ with both radii depending on γ. However, using the Bethe–Ansatz exact solution of the staggered model (see section 3), we find that the continuum limit consists of two free bosons with one radius depending on γ, and the other radius independent of γ. It seems then that the bosonization approach misses an important effect due to the anti-Hermitian term in H.

3. Bethe–Ansatz solution

In this section, we present the solution by Bethe–Ansatz of the model presented in section 2.4. We find that the Bethe roots form two *coupled* Fermi seas, and the elementary excitations are holes close to the Fermi levels. In the continuum limit, we obtain the dressed momentum and energy (3.22) of the holes, and the dressed scattering amplitudes (3.27) between them. The central charge of the theory is $c = 2$. Using the Wiener–Hopf technique for the computation of finite-size corrections (see appendix B), we derive the conformal spectrum (3.31). It has the form of a two-component Coulomb gas.

3.1. Transfer matrix of the vertex model

To find the eigenvalues of the Hamiltonian H (2.13), we use the six-vertex formulation of the TL algebra, which is defined by $\check{R}$-matrix (2.3) and expression (2.16) of the TL generators as 4×4 matrices. To construct the transfer matrix, we introduce the R-matrix $R_{j,j'}(u) \equiv P_{j,j'}\check{R}_{j,j'}(u)$, where $P_{j,j'}$ is the permutation operator between the sites j and j'. The one-row transfer matrix $t(u)$ with twisted boundary conditions is then given by

$$t(u) = \mathrm{Tr}_0 \left[\mathrm{e}^{\mathrm{i}\phi\sigma_0^z} R_{01}(u) R_{02}\left(u - \frac{\pi}{2}\right) \ldots R_{0,2N-1}(u) R_{0,2N}\left(u - \frac{\pi}{2}\right) \right].$$

The two-row transfer matrix is given by $\mathcal{T}(u) = t(u)t\left(u + \frac{\pi}{2}\right)$.

3.2. Bethe–Ansatz equations

Throughout this paper we consider only periodic (twisted) conditions, and hence the total magnetization is always conserved. As a consequence, we can use the algebraic Bethe–Ansatz [1], with the reference state $|0\rangle = |\uparrow \ldots \uparrow\rangle$. Defining the shifted Bethe roots as $\alpha_j \equiv \mathrm{i}(\gamma - 2u_j)$, the BAE and the eigenvalues and eigenvectors of the transfer matrix $t(u)$ in the r-particle sector are

$$\left[\frac{\sinh(\alpha_j - \mathrm{i}\gamma)}{\sinh(\alpha_j + \mathrm{i}\gamma)}\right]^N = -\mathrm{e}^{2\mathrm{i}\phi} \prod_{l=1}^{r} \frac{\sinh\frac{1}{2}(\alpha_j - \alpha_l - 2\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha_j - \alpha_l + 2\mathrm{i}\gamma)}, \tag{3.1}$$

$$\Lambda(u) = \frac{1}{2^N} \left\{ \mathrm{e}^{\mathrm{i}\phi}[\sin 2(\gamma - u)]^N \prod_{j=1}^{r} \frac{\sinh\frac{1}{2}[\mathrm{i}\gamma + (\alpha_j + 2\mathrm{i}u)]}{\sinh\frac{1}{2}[\mathrm{i}\gamma - (\alpha_j + 2\mathrm{i}u)]} \right.$$

$$\left. + \mathrm{e}^{-\mathrm{i}\phi}(-\sin 2u)^N \prod_{j=1}^{r} \frac{\sinh\frac{1}{2}[\mathrm{i}\gamma - (\alpha_j + 2\mathrm{i}u - 2\mathrm{i}\gamma)]}{\sinh\frac{1}{2}[\mathrm{i}\gamma + (\alpha_j + 2\mathrm{i}u - 2\mathrm{i}\gamma)]} \right\}, \tag{3.2}$$

$$|\Psi(u_1, \ldots, u_r)\rangle = B(u_1) \ldots B(u_r)|0\rangle. \tag{3.3}$$

In equation (3.3), we have used the notations of [1] for the monodromy matrix elements. The Bethe states $|\Psi(u_1, \ldots, u_r)\rangle$ are invariant under the two-site cyclic translation $\mathrm{e}^{-2\mathrm{i}P}$. In the very anisotropic limit $u \to 0$, the transfer matrix becomes

$$t(0)\, t\left(\frac{\pi}{2}\right) = \left(-\frac{\sin^2 2\gamma}{4}\right)^N \mathrm{e}^{-2\mathrm{i}P}, \tag{3.4}$$

and the corresponding eigenvalue is, from (3.2),

$$\Lambda(0)\, \Lambda\left(\frac{\pi}{2}\right) = \mathrm{e}^{2\mathrm{i}\phi} \left(-\frac{\sin^2 2\gamma}{4}\right)^N \prod_{j=1}^{r} \frac{\sinh(\alpha_j + \mathrm{i}\gamma)}{\sinh(\alpha_j - \mathrm{i}\gamma)}. \tag{3.5}$$

The energy for the Hamiltonian (2.13) is the logarithmic derivative of the eigenvalue:

$$\begin{aligned} E &\equiv -\frac{1}{2}\sin 2\gamma \; \frac{\mathrm{d}\log[\Lambda(u)\Lambda(u + \pi/2)]}{\mathrm{d}u}\bigg|_{u=0} \\ &= 2N\cos 2\gamma - \sum_{j=1}^{r} \frac{2\sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}. \end{aligned} \tag{3.6}$$

Equations (3.5) and (3.6) show that each Bethe root α contributes to the total momentum and energy by

$$2k_j = -\mathrm{i}\log\frac{\sinh(\alpha_j - \mathrm{i}\gamma)}{\sinh(\alpha_j + \mathrm{i}\gamma)}, \qquad \epsilon_j = -\frac{2\sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}. \tag{3.7}$$

Because of the periodicity property of the Boltzmann weights $\check{R}(u + \pi) = -\check{R}(u)$, the Bethe states are unchanged under $\alpha \to \alpha + 2\mathrm{i}\pi$ for any of the Bethe roots. So we can restrict our study to the strip $0 \leqslant \operatorname{Im}\alpha < 2\pi$. The root α gives a negative contribution to the energy (3.6) if it is of the form

$$\alpha_j^{(0)} = \lambda_j^{(0)} \qquad \text{or} \qquad \alpha_j^{(1)} = \lambda_j^{(1)} + \mathrm{i}\pi, \tag{3.8}$$

with $\lambda_j^{(a)}$ real. So, at low energies, the system is described by two coupled Fermi seas $\{\lambda_j^{(0)}\}, \{\lambda_j^{(1)}\}$. The BAE for the $\lambda_j^{(a)}$ are

$$2Nk\left(\lambda_j^{(a)}\right) = 2\pi I_j^{(a)} + 2\phi - \sum_{b=0,1} \sum_{l=1}^{r^{(b)}} \Theta^{(a-b)}\left(\lambda_j^{(a)} - \lambda_l^{(b)}\right), \tag{3.9}$$

where $r^{(a)}$ is the number of roots $\alpha^{(a)}$. The momentum, energy and scattering phases are given by

$$\begin{aligned} 2k(\lambda) &= -\mathrm{i}\log\frac{\sinh(\mathrm{i}\gamma - \lambda)}{\sinh(\mathrm{i}\gamma + \lambda)}, \\[4pt] \epsilon(\lambda) &= -\sin 2\gamma \times 2k'(\lambda) = -\frac{2\sin^2 2\gamma}{\cosh 2\lambda - \cos 2\gamma}, \\[4pt] \Theta^{(0)}(\lambda) &= -\mathrm{i}\log\frac{\sinh(\mathrm{i}\gamma + \frac{\lambda}{2})}{\sinh(\mathrm{i}\gamma - \frac{\lambda}{2})}, \\[4pt] \Theta^{(\pm 1)}(\lambda) &= -\mathrm{i}\log\frac{\cosh(\mathrm{i}\gamma + \frac{\lambda}{2})}{\cosh(\mathrm{i}\gamma - \frac{\lambda}{2})}. \end{aligned} \tag{3.10}$$

The Bethe integers satisfy $I_j^{(a)} \in \frac{1}{2}(N + r^{(a)} - 1) + \mathbb{Z}$. The total momentum and energy are

$$2Q = \frac{2\pi}{N} \sum_{a=0,1} \sum_{j=0}^{r^{(a)}} \left(I_j^{(a)} + \frac{\phi}{\pi} \right) + \pi(r^{(0)} + r^{(1)}), \tag{3.11}$$

$$E = 2N \cos 2\gamma + \sum_{a=0,1} \sum_{j=1}^{r^{(a)}} \epsilon(\lambda_j^{(a)}). \tag{3.12}$$

A special case is when the Bethe roots $\{\lambda_j^{(0)}\}$, $\{\lambda_j^{(1)}\}$ are identical on the two lines: we call these *symmetric states*. It is a remarkable fact that this subset of the spectrum is given exactly by the Bethe states of the *XXZ* spin chain on a periodic lattice with N sites:

$$H_{XXZ} = -\frac{1}{2} \sum_{m=1}^{N} \left[\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta_0 \, \sigma_m^z \sigma_{m+1}^z \right], \tag{3.13}$$

with $\Delta_0 = -\cos 2\gamma$. Indeed, we have the identities

$$2\widetilde{k} = k_{XXZ}, \qquad \epsilon = \epsilon_{XXZ}, \qquad \Theta^{(0)} + \Theta^{(\pm 1)} = \Theta_{XXZ}, \tag{3.14}$$

where quantities with the subscript '*XXZ*' are related to the Bethe–Ansatz for *XXZ*. Thus, BAE (3.9) for symmetric states is equivalent to the *XXZ* ones, and the energies are related by $E = 2E_{XXZ}$.

Finally, we note that the generic BAE (3.1) are identical to those of an *XXZ* spin chain on a lattice with N sites, with anti-periodic boundary conditions [31]. This equivalence is obtained by substituting $v_j \to 2\alpha_j + \mathrm{i}\pi$ and $\lambda \to 2\mathrm{i}\gamma$ into equation (1.9) of [31], and choosing an appropriate twist. However, the major distinction between the two models is that the *XXZ* chain with anti-periodic boundary conditions does not enjoy total magnetization conservation, and thus the number of Bethe roots for this problem is fixed to $r = N$.

3.3. Continuum limit

The ground state corresponds to $r^{(0)} = r^{(1)} = N/2$, with the Bethe integer distribution (see figure 6(*a*)):

$$\begin{aligned} \{I_j^{(0)}\} &= -\frac{r^{(0)} - 1}{2}, \dots, \frac{r^{(0)} - 1}{2} \\ \{I_j^{(1)}\} &= -\frac{r^{(1)} - 1}{2}, \dots, \frac{r^{(1)} - 1}{2}. \end{aligned} \tag{3.15}$$

The continuum limit is defined as

$$N \to \infty, \qquad r^{(a)}/N \to 1/2. \tag{3.16}$$

In this limit, we assume that the spacing between Bethe roots scales like $1/N$, and we describe the Bethe root distribution by the densities:

$$\rho^{(a)}(\lambda_j^{(a)}) = \frac{1}{N(\lambda_{j+1}^{(a)} - \lambda_j^{(a)})}.$$

We denote $[-C^{(a)}, B^{(a)}]$ the interval spanned by the roots $\lambda_j^{(a)}$. BAE (3.9) become Lieb equations for the root densities:

$$2k'(\lambda) = 2\pi \rho^{(a)}(\lambda) - \sum_{b=0,1} \int_{-C^{(b)}}^{B^{(b)}} \mathrm{d}\mu \, \rho^{(b)}(\mu) \, K^{(a-b)}(\lambda - \mu), \tag{3.17}$$
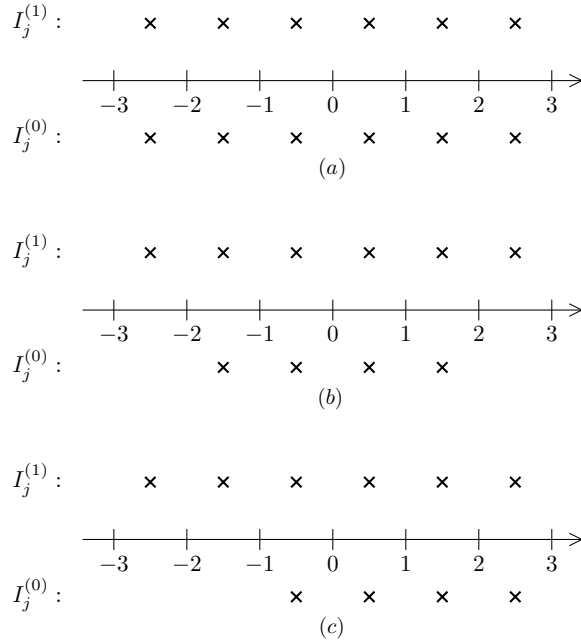
Figure 6. Example of Bethe integer distributions for $N = 12$: (*a*) ground state; (*b*) magnetic excitation $m^{(0)} = 2$; (*c*) combined magnetic–electric excitation $m^{(0)} = 2$, $e^{(0)} = 1$.

where the kernels are given by $K^{(a)} = (\Theta^{(a)})'$. In the ground state, we have $C^{(0,1)}$, $B^{(0,1)} \to \infty$, so equation (3.17) can be solved by the Fourier transform. The solution involves the symmetric and antisymmetric inverse kernels $J^{(\pm)}$ (see appendix A):

$$1 + \widehat{J}^{(\pm)}(\omega) \equiv \frac{2\pi}{2\pi - [\widehat{K}^{(0)}(\omega) \pm \widehat{K}^{(1)}(\omega)]}. \tag{3.18}$$

The ground-state densities are $\rho^{(0)} = \rho^{(1)} = \rho_\infty$, where

$$\rho_\infty = (\delta + J^{(+)}) \star (2k')/(2\pi) = \frac{1}{4\gamma \cos[\pi\lambda/(2\gamma)]}. \tag{3.19}$$

The symbol $\star$ denotes convolution.

An elementary excitation above the ground state consists in a hole λ_h in the distribution $\{\lambda_j^{(0)}\}$ or $\{\lambda_j^{(1)}\}$, interacting with all the particles in both Fermi seas. Let A be a physical quantity defined as

$$A = \frac{1}{N} \sum_{a=0,1} \sum_{j=1}^{r^{(a)}} \alpha(\lambda_j^{(a)})$$

$$A \to \int_{-C^{(0)}}^{B^{(0)}} d\lambda \, \rho^{(0)}(\lambda)\alpha(\lambda) + \int_{-C^{(1)}}^{B^{(1)}} d\lambda \, \rho^{(1)}(\lambda)\alpha(\lambda). \tag{3.20}$$

In the presence of a hole λ_h, the variation of A with respect to the ground-state value A_0 is given by the dressed quantity α_d (see appendix A):

$$A - A_0 = \frac{1}{N} \alpha_d(\lambda_h), \qquad \alpha_d \equiv -(\delta + J^{(+)}) \star \alpha. \tag{3.21}$$

The momentum and energy of a hole are thus

$$2k_d(\lambda) = -2\text{Atan}\left[\tanh\left(\frac{\pi\lambda}{4\gamma}\right)\right],$$

$$\epsilon_d(\lambda) = \frac{\pi\sin 2\gamma}{2\gamma\cosh[\pi\lambda/(2\gamma)]}. \tag{3.22}$$

In the region $\lambda \to \infty$, the dressed momentum is close to the value $-\pi/2$, and the dispersion relation is linear, with Fermi velocity v:

$$\epsilon_d \simeq -v\left(2k_d + \frac{\pi}{2}\right), \qquad v = \frac{\pi\sin 2\gamma}{2\gamma}. \tag{3.23}$$

Hence, hole excitations are gapless, and the theory is critical.

3.4. Dressed scattering amplitudes

In the presence of holes, the root densities $\rho^{(a)}$ coexist with the densities of holes $\rho_h^{(a)}$. Lieb equations (3.17) become

$$\begin{cases} 2\pi\left(\rho^{(0)} + \rho_h^{(0)}\right) = 2k' + K^{(0)} \star \rho^{(0)} + K^{(-1)} \star \rho^{(1)} \\ 2\pi\left(\rho^{(1)} + \rho_h^{(1)}\right) = 2k' + K^{(1)} \star \rho^{(0)} + K^{(0)} \star \rho^{(1)}. \end{cases} \tag{3.24}$$

These coupled equations can be rewritten in terms of $\rho^{(a)} + \rho_h^{(a)}$ and $\rho_h^{(a)}$:

$$\begin{cases} \rho^{(0)} + \rho_h^{(0)} = \rho_\infty - J^{(0)} \star \rho_h^{(0)} - J^{(-1)} \star \rho_h^{(1)} \\ \rho^{(1)} + \rho_h^{(1)} = \rho_\infty - J^{(1)} \star \rho_h^{(0)} - J^{(0)} \star \rho_h^{(1)}, \end{cases} \tag{3.25}$$

where the kernels $J^{(0)}$, $J^{(\pm 1)}$ are defined as

$$J^{(0)} \equiv \tfrac{1}{2}(J^{(+)} + J^{(-)}), \qquad J^{(\pm 1)} \equiv \tfrac{1}{2}(J^{(+)} - J^{(-)}). \tag{3.26}$$

The Fourier transforms of the kernels $J^{(0)}$, $J^{(\pm 1)}$ are

$$\widehat{J}^{(0)}(\omega) = \frac{\sinh(\pi - 3\gamma)\omega}{2\cosh\gamma\omega\sinh(\pi - 2\gamma)\omega},$$

$$\widehat{J}^{(\pm 1)}(\omega) = -\frac{\sinh\gamma\omega}{2\cosh\gamma\omega\sinh(\pi - 2\gamma)\omega}. \tag{3.27}$$

3.5. Central charge and conformal dimensions

The low-energy spectrum consists of 'electromagnetic' excitations above the ground-state distribution, similar to the *XXZ* case [30]. In the present case, the Bethe integer distributions $\{I_j^{(0)}\}$, $\{I_j^{(1)}\}$ can be chosen independently, as depicted in figures 6(*b*) and (*c*). A magnetic excitation consists in removing $m^{(0)}$ (resp. $m^{(1)}$) roots of type $\alpha_j^{(0)}$ (resp. $\alpha_j^{(1)}$) from the ground state, while keeping the Bethe integer distributions $\{I_j^{(0,1)}\}$ symmetric around zero. An electric excitation consists in shifting all integers $I_j^{(0)}$ (resp. $I_j^{(1)}$) by $e^{(0)}$ (resp. $e^{(1)}$).

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The central charge and critical exponents are obtained from finite-size corrections to the total energy and momentum:

$$E_0 \simeq N e_\infty - v \times \frac{\pi c}{6N}, \tag{3.28}$$

$$E - E_0 \simeq v \times \frac{2\pi(\Delta + \bar\Delta)}{N}, \tag{3.29}$$

$$Q = \frac{2\pi(\Delta - \bar\Delta)}{N}, \tag{3.30}$$

where E_0 is the ground-state energy.

We start by discussing the untwisted case $\phi = 0$. The ground state is a symmetric state, and thus the ground-state energy is twice that of the *XXZ* spin chain (3.13). The latter has a Fermi velocity v and central charge 1. Using equation (3.28), the central charge of the staggered model is then $c = 2$. The finite-size corrections to the energies for the electromagnetic excitations are computed from the Bethe–Ansatz solution in appendix B. They yield the conformal dimensions

$$\Delta_{em,\widetilde{em}} = \frac{1}{8}\left(\frac{e}{\sqrt{2g}} + m\sqrt{2g}\right)^2 + \frac{1}{8}(\widetilde{e} + \widetilde{m})^2,$$

$$\bar\Delta_{em,\widetilde{em}} = \frac{1}{8}\left(\frac{e}{\sqrt{2g}} - m\sqrt{2g}\right)^2 + \frac{1}{8}(\widetilde{e} - \widetilde{m})^2, \tag{3.31}$$

$$g = \frac{\pi - 2\gamma}{2\pi}, \qquad 0 < g < \frac{1}{2},$$

where

$$e = e^{(0)} + e^{(1)}, \qquad m = m^{(0)} + m^{(1)},$$
$$\widetilde{e} = e^{(0)} - e^{(1)}, \qquad \widetilde{m} = m^{(0)} - m^{(1)}. \tag{3.32}$$

When the twist ϕ is not zero, the above exponents are still correct, with the change $e \to e + 2\phi/\pi$. In particular, the staggered Potts model corresponds to a twist $\phi = \gamma = \pi e_0$. The ground state has an electric charge $e = 2e_0$, with exponents $\Delta = \bar\Delta = e_0^2/(4g)$, so the effective central charge is

$$c_{\text{tw}} = 2 - \frac{6e_0^2}{g}, \qquad e_0 = \frac{\gamma}{\pi} = \frac{1}{2} - g. \tag{3.33}$$

3.6. Application to the calculation of critical exponents

In the loop formulation, we obtain the *k*-leg dimensions as follows. For any system size N, the number of legs k must be even, and the conformal dimensions are defined with respect to the twisted ground state:

$$h_k = \bar h_k = \Delta_k - \frac{e_0^2}{4g}. \tag{3.34}$$

The k-leg dimension Δ_k corresponds to a magnetic defect $m = k/2$, with a minimal value for $\widetilde{m}$ and electric charges $e = \widetilde{e} = 0$ (no background charge). There are then two distinct cases:

$$h_k = \begin{cases} \dfrac{gk^2}{16} - \dfrac{e_0^2}{4g} & \text{if } k \equiv 0\ [4] \\[3mm] \dfrac{gk^2}{16} + \dfrac{1}{8} - \dfrac{e_0^2}{4g} & \text{if } k \equiv 2\ [4]. \end{cases} \tag{3.35}$$
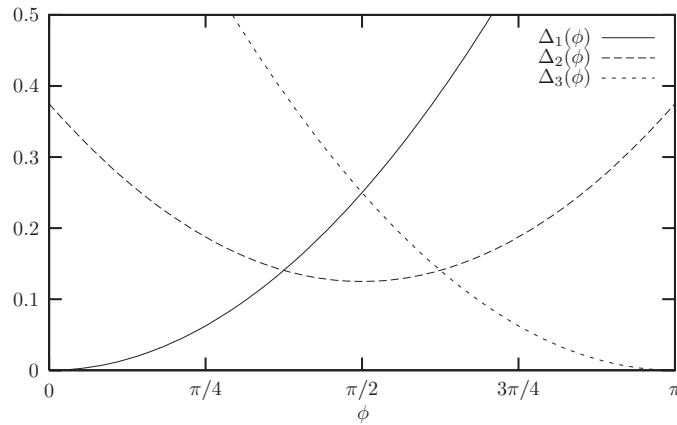
Figure 7. Lowest conformal dimensions in the twisted sector, for $g = 1/4$.

Similarly, the magnetic exponent of the staggered Potts model is defined with respect to the twisted ground state:

$$h_H = \bar{h}_H = \Delta_H - \frac{e_0^2}{4g}. \tag{3.36}$$

The magnetic dimension Δ_H corresponds to a twist $\phi = \pi/2$, which forbids any non-contractible loop around the cylinder. Before we obtain Δ_H, we need to discuss the conformal dimension for the sector $m = \widetilde{m} = 0$ with a general twist ϕ. In the regime $0 < \phi < \pi$, the lowest dimensions are

$$\Delta_1(\phi) = \Delta_{(\phi/\pi,0),(0,0)} = \frac{(\phi/\pi)^2}{4g}, \tag{3.37}$$

$$\Delta_2(\phi) = \Delta_{(\phi/\pi-1,0),(1,0)} = \frac{(\phi/\pi - 1/2)^2}{4g} + \frac{1}{8}, \tag{3.38}$$

$$\Delta_3(\phi) = \Delta_{(\phi/\pi-2,0),(0,0)} = \frac{(1 - \phi/\pi)^2}{4g}. \tag{3.39}$$

The lowest dimension is respectively $\Delta_1, \Delta_2, \Delta_3$ on the intervals $[0, \phi_0]$, $[\phi_0, 1 - \phi_0]$, $[1 - \phi_0, \pi]$, where $\phi_0 = \pi(1 + 2g)/4$. See figure 7. In particular, for $\phi = \pi/2$, we get $\Delta_H = \Delta_2(\pi/2) = 1/8$, and thus

$$h_H = \frac{1}{8} - \frac{e_0^2}{4g}. \tag{3.40}$$

3.7. Numerical checks

First, we check numerically the assumption that the Bethe integers for the ground-state and symmetric magnetic excitations are given by (3.15), even in the presence of a twist: see tables 1 and 2.

　　We also have verified the above expressions for the effective central charge c_{tw} and the k-leg exponents h_k by numerical diagonalization of the transfer matrix for the loop model at the pseudo-isotropic point $u = \frac{\gamma}{2}$.

Table 1. Numerical estimates for the effective central charge in the presence of a twist ϕ. For each value of γ, we give estimates from the numerical diagonalization of H and the numerical solution of the BAE with $N = 8, 10$. These are compared with the exact value $c(\phi) = 2 - 6(\phi/\pi)^2/g$.

ϕ	$\gamma = 0.3$			$\gamma = 0.6$			$\gamma = 0.9$		
	Diagonalization	BAE	Exact	Diagonalization	BAE	Exact	Diagonalization	BAE	Exact
0	2.0897	2.0897	2	2.0706	2.0706	2	2.0524	2.0524	2
$\pi/12$	1.9785	1.9785	1.8970	1.9297	1.9297	1.8652	1.8488	1.8488	1.8049
$\pi/6$	1.6462	1.6462	1.5880	1.5079	1.5079	1.4606	1.2395	1.2395	1.2194
$\pi/4$	1.0964	1.0964	1.0729	0.8086	0.8086	0.7865	0.2287	0.2287	0.2437
$\pi/3$	0.3357	0.3357	0.3519	−0.1629	−0.1629	−0.1574	−1.1761	−1.1761	−1.1223

Table 2. Numerical estimates for the conformal dimension of the sector $S^z = 2$, in the presence of a twist ϕ. For each value of γ, we give estimates from the numerical diagonalization of H and the numerical solution of the BAE with $N = 10$. These are compared with the exact value $X = 2g$.

| ϕ	$\gamma = 0.3$			$\gamma = 0.6$			$\gamma = 0.9$		
	Diagonalization	BAE	Exact	Diagonalization	BAE	exact	Diagonalization	BAE	Exact
0	0.7857	0.7857	0.8090	0.6200	0.6200	0.6180	0.4316	0.4316	0.4270
$\pi/12$	0.7855	0.7855	0.8090	0.6195	0.6195	0.6180	0.4308	0.4308	0.4270
$\pi/6$	0.7850	0.7850	0.8090	0.6179	0.6179	0.6180	0.4284	0.4284	0.4270
$\pi/4$	0.7842	0.7842	0.8090	0.6153	0.6153	0.6180	0.4244	0.4244	0.4270
$\pi/3$	0.7833	0.7833	0.8090	0.6117	0.6117	0.6180	0.4189	0.4189	0.4270
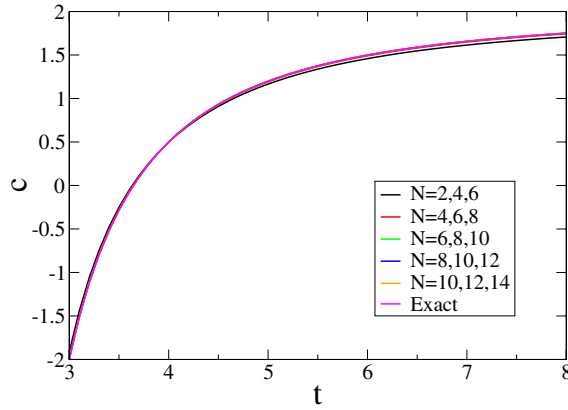
Figure 8. Numerical estimates of the effective central charge c_{tw}, as compared with the exact expression (3.33).
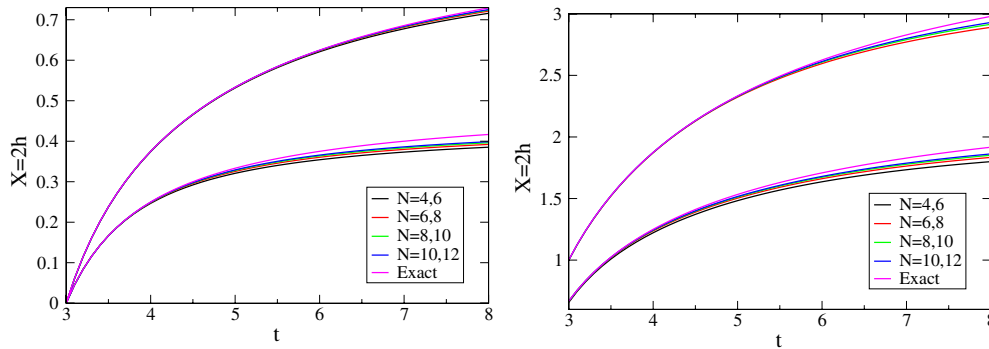

Figure 9. Numerical estimates of the k-leg exponents $X = 2h_k$ with $k = 2, 4, 6, 8$, as compared with the exact expression (3.35).

As usual, the critical exponents can be extracted from the finite-size corrections in N to the dominant eigenvalues in the various sectors labelled by k. We consider the geometry of a strip of width $2N$ strands with periodic boundary conditions in the transverse direction. Estimates for c_{tw} (resp. h_k) are then obtained from fits involving three (resp. two) different sizes N. We use even N throughout. Odd N introduces a twist that leads to different effective exponents that we do not consider any further.

It is convenient to parameterize $\gamma = \frac{\pi}{t}$ through a new parameter t. The results for c_{tw} with $N \leqslant 14$ are shown in figure 8. The agreement with (3.33) is excellent. Results for $X_k = 2h_k$ with $N \leqslant 12$ are given in figure 9 for $k = 2, 4, 6, 8$. The agreement with (3.35) is very satisfactory, especially when $k \equiv 0$ [4].

4. Toroidal partition functions

In this section, we use the results from section 3 to construct explicitly the continuum partition function Z of the statistical model on a torus. Assuming that the conformal spectrum (3.31) obtained from the analysis of the BAE is complete, we sum the conformal characters over

all possible conformal dimensions to obtain Z. The resulting expression (4.14) for Z shows that the continuum limit of the model consists of one boson and two Majorana fermions, which decouple in the bulk and couple only through boundary conditions. We discuss only the untwisted case here, leaving the twisted and Potts model cases (including the study of particular values of Q) to appendix D.

We denote by τ the modular ratio of the torus, and we write $q = \mathrm{e}^{2\mathrm{i}\pi\tau}$. Other notations are defined in appendix C. The primary states of the corresponding CFT have conformal weights $\Delta_{em,\widetilde{e}\widetilde{m}}$ and $\bar{\Delta}_{em,\widetilde{e}\widetilde{m}}$ given by the Bethe–Ansatz results (3.31), where the charges satisfy the parity conditions

$$e + \widetilde{e} \in 2\mathbb{Z}, \qquad m + \widetilde{m} \in 2\mathbb{Z}. \tag{4.1}$$

The partition function on the torus is given by the sum of the generic conformal characters $\chi_\Delta, \bar{\chi}_{\bar{\Delta}}$:

$$Z(g) = \mathrm{Tr}\big(q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24}\big) = \sum_{\Delta,\bar{\Delta}} \chi_\Delta(q)\, \bar{\chi}_{\bar{\Delta}}(\bar{q}), \tag{4.2}$$

where the sum is over all possible primary states, and χ_Δ is the trace of $q^{L_0 - c/24}$ over the descendants of the primary state Φ_Δ:

$$\chi_\Delta(q) = \mathrm{Tr}_\Delta\, q^{L_0 - c/24}. \tag{4.3}$$

The character χ_Δ can be inferred from the possible Bethe integer distributions. Starting from an electromagnetic excitation with dimension Δ, we can create vacancies, by shifting the largest Bethe integer $I_j \to I_j + n,\, n \geqslant 0$. This vacancy state has dimension $\Delta + n$. These vacancies can be combined, and the state with shifts $(n_1, \ldots, n_k)$ has dimension $\Delta + n_1 + \ldots + n_k$. Furthermore, vacancies can be introduced *independently* on the two lines $I_j^{(0)}, I_j^{(1)}$. Let us denote by $p(n)$ the number of partitions of the integer n. We have

$$\chi_\Delta(q) = \sum_{n^{(0)},n^{(1)} \geqslant 0} p(n^{(0)})p(n^{(1)})q^{\Delta + n^{(0)} + n^{(1)} - c/24}$$

$$= \frac{q^{\Delta + (2-c)/24}}{\eta(\tau)^2}, \tag{4.4}$$

where $\eta(\tau)$ is the Dedekind function (C.3). Using (4.4) with $c = 2$ and the parity conditions (4.1), we obtain

$$Z(g) = \frac{1}{|\eta(\tau)|^4} \sum_{\substack{m \equiv \widetilde{m}\,[2] \\ e \equiv \widetilde{e}\,[2]}} q^{\Delta_{em,\widetilde{e}\widetilde{m}}} \bar{q}^{\bar{\Delta}_{em,\widetilde{e}\widetilde{m}}}. \tag{4.5}$$

Using the Poisson summation (C.5), this can be written as

$$Z(g) = 2\Bigg(A \sum_{m,m'\ \text{even}} + B \sum_{m\ \text{even},m'\ \text{odd}}$$

$$+ C \sum_{m\ \text{odd},m'\ \text{even}} + D \sum_{m,m'\ \text{odd}} \Bigg) Z_{m,m'}(g), \tag{4.6}$$

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where

$$A = \sum_{m,m' \text{ even}} Z_{m,m'}(1/2), \qquad B = \sum_{\substack{m \text{ even} \\ m' \text{ odd}}} Z_{m,m'}(1/2),$$

$$C = \sum_{\substack{m \text{ odd} \\ m' \text{ even}}} Z_{m,m'}(1/2), \qquad D = \sum_{m,m' \text{ odd}} Z_{m,m'}(1/2),$$

(4.7)

and $Z_{m,m'}(g)$ is the bosonic partition function with defects m, m' (see (C.4)).

The partition sums A, B, C, D can, in turn, be expressed in terms of the Jacobi ones (C.9), using (C.5) again:

$$A = \tfrac{1}{4}\big(Z_2^2 + Z_3^2 + Z_4^2\big), \qquad B = \tfrac{1}{4}\big(-Z_2^2 + Z_3^2 + Z_4^2\big),$$

$$C = \tfrac{1}{4}\big(Z_2^2 + Z_3^2 - Z_4^2\big), \qquad D = \tfrac{1}{4}\big(Z_2^2 - Z_3^2 + Z_4^2\big).$$

(4.8)

Using the transformation of Jacobi and Coulombic partition functions under modular transformations, one can easily show that expression (4.6) is modular invariant. Let $\mathcal{Z}(r, r')$ be the partition function of the Ising model on a torus with respective boundary conditions on the spins σ in the two directions of the torus:

$$\sigma \to (-1)^r \sigma, \qquad \sigma \to (-1)^{r'} \sigma, \qquad r, r' \in \{0, 1\}^2. \tag{4.9}$$

Using relation (C.10) between $\mathcal{Z}_{r,r'}$ and Z_ν, the partition sums A, B, C, D are written in terms of the $\mathcal{Z}_{r,r'}$:

$$A = \tfrac{1}{4}\big[\mathcal{Z}_{0,0}^2 + \mathcal{Z}_{0,1}^2 + \mathcal{Z}_{1,0}^2 + \mathcal{Z}_{1,1}^2\big], \tag{4.10}$$

$$B = \tfrac{1}{2}[\mathcal{Z}_{0,0}\mathcal{Z}_{0,1} - \mathcal{Z}_{1,0}\mathcal{Z}_{1,1}], \tag{4.11}$$

$$C = \tfrac{1}{2}[\mathcal{Z}_{0,0}\mathcal{Z}_{1,0} - \mathcal{Z}_{0,1}\mathcal{Z}_{1,1}], \tag{4.12}$$

$$D = \tfrac{1}{2}[\mathcal{Z}_{0,0}\mathcal{Z}_{1,1} - \mathcal{Z}_{0,1}\mathcal{Z}_{1,0}]. \tag{4.13}$$

Hence, from (4.6) and (4.10–4.13), the partition function $Z(g)$ reads

$$Z(g) = \frac{1}{2} \sum_{\substack{m \equiv r_1 + r_2 \ [2] \\ m' \equiv r_1' + r_2' \ [2]}} (-1)^{r_1 r_2' + r_1' r_2} \, \mathcal{Z}_{r_1, r_1'} \, \mathcal{Z}_{r_2, r_2'} \, Z_{m,m'}(g). \tag{4.14}$$

The degrees of freedom contained in $Z(g)$ are a compact boson φ (see (C.1)) with coupling constant $g = (\pi - 2\gamma)/(2\pi)$, and two sets of Ising spins σ_1, σ_2. The boundary defects for $\varphi, \sigma_1, \sigma_2$ are respectively (m, m'), (r_1, r_1'), (r_2, r_2'), and obey parity conditions, as shown in (4.14). Apart from these conditions, the three degrees of freedom $\varphi, \sigma_1, \sigma_2$ are decoupled. These results are very similar to what was found in [32] for a lattice model related to $N = 1$ superconformal theories, where only one Ising spin was present.

Like it was done in [32] for the 19-vertex model, here we can also identify the degrees of freedom $\varphi, \sigma_1, \sigma_2$ in the lattice model. For this purpose, we consider the vertex model defined by the block $\check{\mathcal{R}}$-matrix (see figure 4). It was shown in [5] that there are 38 possible vertices. Each edge can be in one of four states: $\uparrow, \downarrow, |, \|$. Let $N_\alpha(\mathbf{r})$ be the number of edges adjacent to the site $\mathbf{r}$, which are in the state α. An essential property of the model, arising from the combination of the magnetization conservation and $\mathbb{Z}/2\mathbb{Z}$ symmetry, is that $N_|$ and $N_\|$ are both even for every vertex. Thus, for a given lattice configuration, the lines formed by the

| and ‖ edges can be viewed as the domain walls of two distinct Ising models, both living on the dual lattice. The remaining edges carry arrows, which define a height (SOS) model on the dual lattice. Although these three degrees of freedom are coupled in the lattice model, our results on the continuum partition function show that they decouple in the continuum limit, except for their boundary conditions, which keep track of the parity of domain walls and arrows around each direction of the torus.

5. Integrable massive deformation

In this section, we follow the approach of [13] to construct a massive deformation of the lattice model, and study its excitation spectrum. Using the dressed scattering amplitudes, we obtain partly the *S*-matrix for elementary excitations. We then complete this result with some natural conjecture, based on symmetry and unitarity. We further use the known results from [14] on this *S*-matrix to conjecture a TBA diagram, and use the TBA equations to calculate the ground-state energy scaling function. In the UV limit, we retrieve the results from section 3. Finally, we use these results to propose an effective QFT for the massive deformation, which is a complex version of the $C_2^{(2)}$ Toda theory.

5.1. Massive integrable deformation on the lattice

We now consider a deformation of our model where the spectral parameters acquire an extra staggering, this time in the imaginary direction. We choose the pattern $u + i\Lambda/2$, $u - i\Lambda/2$, $u + \pi/2 + i\Lambda/2$, $u + \pi/2 - i\Lambda/2$. This kind of construction has been widely used to induce an integrable massive deformation from integrable lattice models [13, 34]. We obtain a modified set of Bethe equations:

$$\begin{cases} 2\pi(\rho + \rho_h)^{(0)} = 2k'_\Lambda + K^{(0)} \star \rho^{(0)} + K^{(-1)} \star \rho^{(1)} \\ 2\pi(\rho + \rho_h)^{(1)} = 2k'_\Lambda + K^{(1)} \star \rho^{(0)} + K^{(0)} \star \rho^{(1)}, \end{cases} \tag{5.1}$$

where $k_\Lambda(\lambda) = k(\lambda + \Lambda) + k(\lambda - \Lambda)$. To explore the corresponding physics, we write what is often called the physical equations, that is, the equations describing scattering of dressed excitations. Since the ground state is obtained by filling up the $\rho^{(0)}$ and $\rho^{(1)}$ lines, this is easily done by re-expressing the equations so that the densities of holes appear on the right-hand side (see section 3.4). We find

$$\begin{cases} 2\pi(\rho + \rho_h)^{(0)} = s + \Phi^{(0,0)} \star \rho_h^{(0)} + \Phi^{(0,1)} \star \rho_h^{(1)} \\ 2\pi(\rho + \rho_h)^{(1)} = s + \Phi^{(1,0)} \star \rho_h^{(0)} + \Phi^{(1,1)} \star \rho_h^{(1)}, \end{cases} \tag{5.2}$$

where $\Phi^{(a,b)} = -2\pi J^{(a-b)}$, and

$$s(\lambda) = \frac{\pi}{2\gamma} \left[\frac{1}{\cosh \frac{\pi}{2\gamma}(\lambda - \Lambda)} + \frac{1}{\cosh \frac{\pi}{2\gamma}(\lambda + \Lambda)} \right]. \tag{5.3}$$

This function has tails at $|\lambda| \gg \Lambda$ where $s(\lambda)$ decays exponentially as in the massless case. These describe a 'ghost' of the initial massless theory, whose physics does not depend on Λ, and which we will not discuss in the following. It decouples entirely from the region where $|\lambda| \ll \Lambda$, which is of interest to us. In this region, we have

$$s(\lambda) \approx \frac{2\pi}{\gamma} e^{-\pi\Lambda/(2\gamma)} \cosh \frac{\pi\lambda}{2\gamma}, \tag{5.4}$$

so the corresponding momentum and energy are

$$2k_d(\lambda) = -\int_0^\lambda s(\mu)\mathrm{d}\mu \approx -4\mathrm{e}^{-\pi\Lambda/(2\gamma)} \sinh\frac{\pi\lambda}{2\gamma}, \tag{5.5}$$

$$\epsilon_d(\lambda) = \sin 2\gamma \ s(\lambda) \approx 4v \ \mathrm{e}^{-\pi\Lambda/(2\gamma)} \cosh\frac{\pi\lambda}{2\gamma}. \tag{5.6}$$

We thus obtain a massive relativistic spectrum, as happens systematically in this kind of construction. The mass is given by

$$\mu = 4\exp\left(-\frac{\pi\Lambda}{2\gamma}\right). \tag{5.7}$$

The question is then, what kind of scattering theory do we obtain, and what quantum field theory does it correspond to?

5.2. Scattering theory

To answer the above question, we start by reinterpreting the kernels $\Phi^{(a,b)}$ as derivatives of scattering phases between basic particles. We will now denote the holes in the 0, 1 sea by the labels $0-$ and $1-$. If we rescale the parameters λ, ω to

$$\theta = \frac{\pi}{2\gamma} \times \lambda, \qquad k = \frac{2\gamma}{\pi} \times \omega, \tag{5.8}$$

and set $t = \pi/\gamma$, we obtain, up to a constant phase which will be obtained below:

$$\begin{aligned}
S_{--}^{(0,0)}(\theta) &= S_{--}^{(1,1)}(\theta) \propto \exp\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}k}{k}\frac{\sin k\theta \ \sinh\frac{k\pi(t-3)}{2}}{\sinh\frac{k\pi(t-2)}{2}\cosh\frac{k\pi}{2}}, \\
S_{--}^{(0,1)}(\theta) &= S_{--}^{(1,0)}(\theta) \propto \mathrm{i}\exp-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}k}{k}\frac{\sin k\theta \ \sinh\frac{k\pi}{2}}{\sinh\frac{k\pi(t-2)}{2}\cosh\frac{k\pi}{2}}.
\end{aligned} \tag{5.9}$$

These two S-matrix elements can be interpreted in terms of the scattering matrix $\mathcal{S}_{ij}(\beta_{\mathrm{SG}};\theta)$ of the sine-Gordon (SG) model [35], with action

$$A_{\mathrm{SG}}[\varphi] = \int\left[\frac{1}{2}\partial_\nu\varphi\partial_\nu\varphi + \frac{\mu_0^2}{\beta_{\mathrm{SG}}^2}\cos(\beta_{\mathrm{SG}}\varphi)\right]. \tag{5.10}$$

If we set

$$\frac{\beta_{\mathrm{SG}}^2}{8\pi} = \frac{t-2}{t-1}, \tag{5.11}$$

then $S_{--}^{(0,0)}$ is the kink–kink (or antikink–antikink) scattering element for the SG model [35]:

$$S_{--}^{(0,0)}(\theta) = S_{--}^{(1,1)}(\theta) = \mathcal{S}_{--}(\beta_{\mathrm{SG}};\theta). \tag{5.12}$$

It is natural to identify the holes as two types of antikinks $(0, 1)$. We expect—but this remains for now a conjecture—the scattering theory to contain also a corresponding doublet of kinks, with a full scattering within the $(0, 0)$ and $(1, 1)$ sectors described by two copies of the sine-Gordon S-matrix.

We now observe that the kernels $S_{--}^{(0,1)}$, $S_{--}^{(1,0)}$ are related to the SG scattering matrix $\widetilde{\mathcal{S}}$ with an imaginary shift in the rapidity θ [14]. The scattering theory defined by $S^{(0,0)} = S^{(1,1)} = \mathcal{S}$, $S^{(0,1)} = S^{(1,0)} = \widetilde{\mathcal{S}}$ was introduced in [14], where it was proposed as the scattering theory for left/right (L/R) massless particles describing the flow between minimal models of CFT under a perturbation by the Φ_{13} primary operator. In [14], using the

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unitarity and crossing conditions, the normalizing factors for the *S*-matrix were computed. The resulting scattering theory is as follows:

$$\text{four basic particles}: 0+, 0-, 1+, 1-$$

$$S^{(0,0)}(\theta) = S^{(1,1)}(\theta) = \mathcal{S}(\beta_{\text{SG}}; \theta)$$

$$S^{(0,1)}(\theta) = \frac{\widetilde{Z}(\theta)}{Z\left(\theta + i\pi \frac{t-2}{2}\right)} \, \mathcal{S}\left(\beta_{\text{SG}}; \theta + i\pi \frac{t-2}{2}\right)$$

$$S^{(1,0)}(\theta) = \frac{-\widetilde{Z}(\theta)}{Z\left(\theta - i\pi \frac{t-2}{2}\right)} \, \mathcal{S}\left(\beta_{\text{SG}}; \theta - i\pi \frac{t-2}{2}\right), \tag{5.13}$$

where the normalization factors read

$$Z(\theta) = \frac{1}{\sinh \frac{i\pi - \theta}{t-2}} \, \exp \frac{i}{2} \int \frac{dk}{k} \frac{\sin k\theta \, \sinh \frac{k\pi}{2}(t-3)}{\sinh \frac{k\pi}{2}(t-2) \cosh \frac{k\pi}{2}},$$

$$\widetilde{Z}(\theta) = \frac{1}{\cosh \frac{i\pi - \theta}{t-2}} \, \exp -\frac{i}{2} \int \frac{dk}{k} \frac{\sin k\theta \, \sinh \frac{k\pi}{2}}{\sinh \frac{k\pi}{2}(t-2) \cosh \frac{k\pi}{2}}. \tag{5.14}$$

From (5.11), we see that the SG *S*-matrices are in the attractive regime for $t \in [2, 3]$ and repulsive regime otherwise. We stress that 0, 1 are not antiparticles of each other.

5.3. Ground-state energy

The scaling function for the ground-state energy is the relevant object to describe the RG flow of a scattering theory. We consider the system on a finite circle of circumference *R*. Then the ground-state energy $E(\mu, R)$ has the scaling form:

$$E(\mu, R) = \frac{2\pi}{R} F(\mu R), \tag{5.15}$$

where μ is the mass of the elementary particles, given in (5.7).

In the present case, the ground-state energy can be obtained simply, using the following identity on the dressed kernels:

$$\hat{\Phi}^{(0,0)}(\omega) + \hat{\Phi}^{(0,1)}(\omega) = \frac{2\pi \sinh(\pi - 4\gamma)\omega/2}{2 \cosh \gamma \omega \sinh(\pi - 2\gamma)\omega/2}. \tag{5.16}$$

The right-hand side of (5.16) is exactly, in terms of the same rapidity θ, the sine-Gordon kernel but for yet another value of the coupling, given by

$$\frac{\widetilde{\beta}_{\text{SG}}^2}{8\pi} = \frac{t-2}{t}. \tag{5.17}$$

In other words,

$$S^{00}_{--}(\theta) \, S^{01}_{--}(\theta) = \mathcal{S}_{--}(\widetilde{\beta}_{\text{SG}}; \theta). \tag{5.18}$$

Assuming that the symmetry is not broken between the two types of roots in the ground state, it follows immediately that the ground-state energy (calculated, e.g., by the method of [34]) is twice the ground-state energy of the sine-Gordon model with the same mass for the kinks, and at this renormalized value of the coupling:

$$E(\mu, R) = 2\mathcal{E}(\widetilde{\beta}_{\text{SG}}; \mu, R). \tag{5.19}$$

Figure 10. The TBA diagram for the RSOS restrictions of our theory. The leftmost and rightmost nodes are both massive.

This result is in fact quite expected if we recall that symmetric solutions to the Bethe equations satisfy precisely the same system as in the *XXZ* chain whose staggering produces the sine-Gordon theory in the continuum limit[5]. Of course, this result could only be completely established by a study of the complex root solutions of the Bethe equations, which we are not doing at this stage.

It would be more satisfactory to establish result (5.19) directly from the scattering theory. This is difficult, because it requires understanding of the complex solutions of the Bethe equations relevant to the thermodynamics, and these solutions seem to exhibit some unheard of features in general. We can nevertheless make progress on the RSOS version of the model (for *t* integer). The ground-state energy $E(\mu, R)$ is generally obtained by the TBA for relativistic scattering theories, introduced in [36]. The idea of the method is to consider a Euclidean theory on a semi-infinite cylinder of dimensions $R \times L$, and to write the partition function in two ways:

$$Z(R, L) = \exp[-E(\mu, R)L] = \mathrm{Tr}(e^{-RH_L}), \tag{5.20}$$

where H_L is the Hamiltonian on an infinite domain when $L \to \infty$. The problem of computing $E(\mu, R)$ is thus equivalent to the computation of the free energy on an infinite domain, at finite temperature $1/R$. So one has to find the density of elementary particles which satisfies BAE (5.2), and maximizes the free energy at temperature $1/R$. This results in the nonlinear integral equations and the ground-state energy, given in terms of the pseudo-energies ϵ_a [36]:

$$\epsilon_a(\theta) = \mu_a R \cosh \theta - \sum_b (\phi_{ab} \star L_b)(\theta),$$

$$L_a = \log(1 + e^{-\epsilon_a}), \qquad \phi_{ab} = \frac{N_{ab}}{2\pi \cosh \theta}, \tag{5.21}$$

$$E(\mu, R) = -\sum_a \frac{\mu_a}{2\pi} \int L_a(\theta) \cosh \theta \, \mathrm{d}\theta,$$

where μ_a is the mass of particles of type a, and N_{ab} is the adjacency matrix of a diagram describing the scattering between particles. In a diagonal (non-reflecting) scattering theory, the μ_a and ϕ_{ab} would be given directly from the dispersion relations and the *S*-matrix for elementary excitations. However, the present model does allow reflection of the particles. The main difficulty here is then to find the correct TBA diagram and masses for the *S*-matrix we want to study. Following the ideas of [14–16] (see the introduction), we conjecture that the TBA diagram for the scattering theory (5.13) with mass μ (5.7) is the diagram of figure 10.

Now, assuming the above conjecture is correct, we check that the TBA equations (5.21) for the diagram of figure 10 lead to the central charge (3.33) in the UV limit $R \to 0$. As in [15], the ground-state energy in the UV limit is obtained in terms of the limiting values of ϵ_a

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[5] The careful reader might worry about the role of Λ in both points of view. The staggering in the equivalent *XXZ* system involves 2Λ, but the anisotropy is also doubled, so the physical mass remains the same.

in the UV and IR limit:

$$E(\mu, R) \simeq -\frac{1}{\pi R} \left[\sum_{a=1}^{t-3} \mathcal{L}\left(\frac{x_a}{1+x_a}\right) - \sum_{a=1}^{t-3} \mathcal{L}\left(\frac{y_a}{1+y_a}\right) \right],$$

$$x_a = \lim_{R\to 0}[e^{-\epsilon_a(0)}], \qquad y_a = \lim_{R\to\infty}[e^{-\epsilon_a(0)}], \tag{5.22}$$

where $\mathcal{L}$ is the Rogers dilogarithm:

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \left[\frac{\log t}{1-t} + \frac{\log(1-t)}{t} \right]. \tag{5.23}$$

The quantities x_a, y_a are determined by the adjacency matrix N_{ab} and the masses μ_a [15]:

$$x_a^2 = \prod_{b=1}^{t-3}(1+x_b)^{N_{ab}}, \qquad y_a^2 = \prod_{b|\mu_b=0}(1+y_b)^{N_{ab}}. \tag{5.24}$$

To connect this with known results [15] on the RSOS central charge, we introduce the quantities z_a which satisfy

$$z_a^2 = \prod_{b=2}^{t-3}(1+z_b)^{N_{ab}}, \qquad a = 2, \ldots, t-3, \tag{5.25}$$

and write (5.22) as

$$E(\mu, R) \simeq -\frac{1}{\pi R} \left[\sum_{a=1}^{t-3} \mathcal{L}\left(\frac{x_a}{1+x_a}\right) - \sum_{a=2}^{t-3} \mathcal{L}\left(\frac{z_a}{1+z_a}\right) \right]$$

$$-\frac{1}{\pi R} \left[\sum_{a=2}^{t-3} \mathcal{L}\left(\frac{z_a}{1+z_a}\right) - \sum_{a=2}^{t-4} \mathcal{L}\left(\frac{y_a}{1+y_a}\right) \right]. \tag{5.26}$$

The above expression is exactly the sum of ground-state energies for the A_{t-2} and A_{t-3} RSOS models, so the central charge is

$$c = \left[1 - \frac{6}{t(t-1)} \right] + \left[1 - \frac{6}{(t-1)(t-2)} \right] = 2 - \frac{12}{t(t-2)}, \tag{5.27}$$

which is the central charge (3.33) of the critical theory.

Finally, we show that, throughout the scaling regime, the ground-state energy $E(\mu, R)$ is twice that of the corresponding twisted sine-Gordon model. Different cases arise, and we will discuss only one: the case when $t - 3 = 2n + 1, n \in \mathbb{N}$. We can then relabel the nodes on the diagram of figure 10, so that the n leftmost ones are called $1, \ldots, n$, the n rightmost ones $\bar{n}, \ldots, \bar{1}$ and the middle one is 0. The TBA equations (5.21) then read

$$\epsilon_a = \delta_{a1}\mu R \cosh\theta - \sum_b N_{ab}\,\phi \star L_b, \qquad a = 1, \ldots, n-1,$$

$$\epsilon_{\bar{a}} = \delta_{\bar{a}\bar{1}}\mu R \cosh\theta - \sum_b N_{ab}\,\phi \star L_{\bar{b}}, \qquad a = 1, \ldots, n-1,$$

$$\epsilon_n = -\phi \star (L_{n-1} + L_0),$$

$$\epsilon_{\bar{n}} = -\phi \star (L_{\overline{n-1}} + L_0),$$

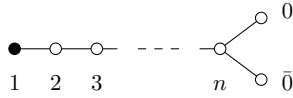$$\epsilon_0 = -\phi \star (L_n + L_{\bar{n}}), \tag{5.28}$$

Figure 11. The TBA for the sine-Gordon model at coupling $\widetilde{\beta}_{SG}^2/(8\pi) = (n+1)/(n+2)$.

where we have set $\phi(\theta) = 1/(2\pi \cosh\theta)$. We consider symmetric solutions under the exchange of a and $\bar{a}$:

$$\epsilon_a = \delta_{a1}\mu R \cosh\theta - \sum_b N_{ab}\phi \star L_b, \qquad a = 1, \ldots, n-1,$$

$$\epsilon_n = -\phi \star L_{n-1} - \phi \star \log(1 + \mathrm{i}\, \mathrm{e}^{-\epsilon_0/2}) - \phi \star \log(1 - \mathrm{i}\, \mathrm{e}^{-\epsilon_0/2}),$$

$$\frac{\epsilon_0}{2} = -\phi \star L_n,$$

$$\epsilon_{\bar{a}} = \epsilon_a. \tag{5.29}$$

The ground-state energy is meanwhile

$$E(\mu, R) = -2 \sum_{a=1}^{n} \frac{\mu_a}{2\pi} \int L_a(\theta) \cosh\theta \, \mathrm{d}\theta. \tag{5.30}$$

We thus see that our system has twice the ground-state energy of a TBA whose diagram is as in figure 11, and which involves a fugacity for the two end nodes of the fork equal to $\pm\mathrm{i}$. This is exactly the TBA for the twisted sine-Gordon model, following the lines of [37]. For this value of the twist in particular, the results in [37] give the central charge (using equation (21) of [37], with $t + 1 = n + 2$ the total number of nodes; $\lambda_{1^-} = -\lambda_{t-1} = \mathrm{i}$ in equation (20) of [37]):

$$\widetilde{c} = 1 - \frac{3/2}{(n+1)(n+2)}. \tag{5.31}$$

Comparing (5.27) and (5.31) for $t - 3 = 2n + 1$, we see that $c = 2\widetilde{c}$.

5.4. The field theory

We now try to identify the field theory described by our TBA. This of course involves a bit of guesswork.

First, recall we have found that the physical mass of the theory scales as $\mu \propto \mathrm{e}^{-t\Lambda/2}$ where $\gamma = \pi/t$. On the other hand, we can in general expect that we are facing the perturbation of a model of central charge $c = 2$ by some operator Φ_h of conformal dimension h, with action

$$A = A_{\mathrm{CFT}} + \omega \int \mathrm{d}^2x \, \Phi_h, \tag{5.32}$$

where A_{CFT} is the action for the critical UV limit. The dimension of the coupling constant ω is $[\omega] = R^{2h-2}$, and thus the mass in the TBA scales as $\mu \propto \omega^{1/(2-2h)}$. A detailed look at the microscopic Hamiltonian shows that the bare coupling is proportional to $\mathrm{e}^{-\Lambda}$. If follows that $2 - 2h = 2/t$, and thus

$$h = \frac{t-1}{t}. \tag{5.33}$$

In general, the TBA approach will give for the ground-state energy $E(\mu, R)$ a series in μ^α, where the exponent α is given by $\alpha = 2(1 - h)$ if correlations of Φ_h are non-zero

(*e.g.* perturbation by Φ_{13} in minimal models), $\alpha = 4(1 - h)$ if only even correlators are non-zero (e.g. the sine-Gordon model). Other possibilities exist, e.g. $\alpha = 8(1 - h)$ if only correlators involving a number of operators *multiple of four* are non-zero.

Since the ground-state energy of our model is twice that of the SG model at $\widetilde{\beta}_{SG}^2/(8\pi) = (t - 2)/t$, it follows that $\alpha = 4(1 - h_{SG})$, where $h_{SG} = (t - 2)/t$. Meanwhile, we have identified earlier the dimension of the perturbation as $h = (t - 1)/t$, and thus $\alpha = 8(1 - h)$. We are forced to conclude therefore that in our problem indeed *only correlators involving a number of operators multiple of four are non-zero*.

Meanwhile, the structure of the scattering matrix suggests the same quantum group symmetry as the one in the β_{SG} theory, with, for generic values of t, only one conserved charge, since the S^{01} elements allow reflection of charges between 0 and 1 sectors (this does not occur at the special points where $t/(t - 2)$ is an integer). Finally, the structure of finite-size effects showed that the CFT was made of a Dirac fermion and a boson of t-dependent radius. This all leads us to propose that the action of the theory is

$$A[\varphi, \psi_1, \psi_2] = \int \left[\frac{1}{2} \partial_\nu \varphi \partial_\nu \varphi + \mathrm{i}(\bar{\psi}_1 \slashed{\partial} \psi_1 + \bar{\psi}_2 \slashed{\partial} \psi_2) \right] \mathrm{d}^2 x$$
$$+ \mu_0 \int \left[\psi_1 \bar{\psi}_1 \mathrm{e}^{\mathrm{i}\beta\varphi} + \psi_2 \bar{\psi}_2 \mathrm{e}^{-\mathrm{i}\beta\varphi} \right] \mathrm{d}^2 x, \tag{5.34}$$

where φ is the boson and ψ_1, ψ_2 are the two Majorana components of the Dirac theory. One can check that this theory is indeed integrable using the non-local conserved charges $\psi_1 \mathrm{e}^{-(4\mathrm{i}\pi/\beta)\varphi}$ and $\psi_2 \mathrm{e}^{(4\mathrm{i}\pi/\beta)\varphi}$. The algebra satisfied by these charges leads to $SU(2)_q$ with quantum-group deformation parameter [38]:

$$q = - \exp(-\mathrm{i}\pi/\delta), \qquad \delta = \frac{2\beta^2}{4\pi - \beta^2}. \tag{5.35}$$

Meanwhile, the basic SG S-matrix with the foregoing value of β_{SG} also has the quantum group symmetry [39], with deformation parameter that corresponds to

$$\delta = \frac{\beta_{SG}^2}{8\pi - \beta_{SG}^2} = t - 2. \tag{5.36}$$

By requiring that the symmetry of the S-matrix is the symmetry of the action, we identify the above two expressions for δ, and we get

$$\frac{\beta^2}{8\pi} = \frac{t - 2}{2t}. \tag{5.37}$$

The dimension of the perturbation is indeed

$$h = \frac{1}{2} + \frac{t - 2}{2t} = \frac{t - 1}{t}, \tag{5.38}$$

and clearly only correlators involving a number of operators multiple of four are non-zero. We also obtain a non-unitary theory, which is expected from the presence of complex terms in the Hamiltonian.

An important check of our proposal would be to see if the ground-state energy of theory (5.34) is twice the ground-state energy of the related sine-Gordon model. One might first tackle this question in perturbation theory. We will leave this for future work, and content ourselves by examining the question in the limit $\beta \to 0$. Then $\widetilde{\beta}_{SG} \to 0$ and we expect, on the one hand, the ground-state energy to be twice the one of a free boson. On the other hand, our action reduces naively to two identical massive Majorana fermions. In the limit $\beta \to 0$, however, counter-terms are needed and a φ^2 term also appears (exactly as in the case of $N = 1$

theories [40]), leading to an additional free massive boson. Denote by E_b the ground-state energy of such a boson, and E_f the ground-state energy of a free Majorana fermion:

$$E_b(\mu, R) = -\frac{\mu}{2\pi} \int_{-\infty}^{\infty} \log(1 - e^{-\mu R \cosh \theta}) \cosh \theta \, d\theta,$$

$$E_f(\mu, R) = \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \log(1 + e^{-\mu R \cosh \theta}) \cosh \theta \, d\theta. \tag{5.39}$$

We have the identity

$$2E_b(\mu, R) = E_b(2\mu, R) + 2E_f(\mu, R), \tag{5.40}$$

so we see indeed that the ground-state energy of our field theory will be twice the ground-state energy of the SG model in the limit of vanishing coupling provided that the mass terms are in the proper ratios. More precisely, the left-hand side of (5.40) corresponds to twice the ground-state energy for the theory:

$$A[\varphi] = \int \left(\frac{1}{2} \partial_\nu \varphi \partial_\nu \varphi + \mu^2 \varphi^2 \right), \tag{5.41}$$

so near $\beta_{\text{SG}} = 0$ we will need an action of the form

$$A[\varphi, \psi_1, \psi_2] = \int \left[\frac{1}{2} \partial_\nu \varphi \partial_\nu \varphi + i(\bar\psi_1 \slashed\partial \psi_1 + \bar\psi_2 \slashed\partial \psi_2) \right] d^2 x$$

$$+ \mu_0 \int [\psi_1 \bar\psi_1 e^{i\beta\varphi} + \psi_2 \bar\psi_2 e^{-i\beta\varphi}] d^2 x$$

$$+ \frac{\mu_0^2}{\beta^2} \int \cos(2\beta\varphi) d^2 x. \tag{5.42}$$

We now observe that our theory is identical to the $C_2^{(2)}$ Toda theory (more precisely, we need in fact to set $n = 1$ in the more general $C_{n+1}^{(2)}$ theory whose form is valid for $n > 1$ only) whose Lagrangian would read [41, 42]

$$\mathcal{L}[\varphi, \psi_1, \psi_2] = \frac{1}{2} \partial_\nu \varphi \partial_\nu \varphi + i(\bar\psi_1 \slashed\partial \psi_1 + \bar\psi_2 \slashed\partial \psi_2)$$

$$- \mu_0 \left(\bar\psi_1 \psi_1 e^{g\varphi/\sqrt{2}} + \bar\psi_2 \psi_2 e^{-g\varphi/\sqrt{2}} \right)$$

$$- \frac{2\mu_0^2}{g^2} \cosh(\sqrt{2}g\varphi). \tag{5.43}$$

Clearly we have to set $g = i\sqrt{2}\beta$. We then see that in the limit $\beta \to 0$ the boson has the mass parameter twice the one of the Majorana fermions, in agreement with (5.40). Our results can be summarized as follows.

- The continuum limit of our lattice model is the complex $C_2^{(2)}$ theory (5.42).
- The S-matrix of this theory is given by (5.13), with $\beta_{\text{SG}}^2/(8\pi) = 2\beta^2/(\beta^2 + 4\pi)$.
- The ground-state energy is twice the ground-state energy of the sine-Gordon theory at coupling $\tilde\beta_{\text{SG}}^2 = 2\beta^2$.

6. Conclusion

From the point of view of integrable statistical models, one can think of several ways to generalize the construction of the $\mathbb{Z}_2$ model. First, one can build a model with a staggering of

period $p > 2$, which has a $\mathbb{Z}/p\mathbb{Z}$ symmetry [43]. We guess that the corresponding continuum limit will be related to a product of p copies of $SU(2)_1$, made anisotropic by a $J^3 J^3$ term as in the case $p = 2$. What the integrable massive deformation might be is however more mysterious. Also, the effective field theory for the analogue of the non-compact regime [5] is less clear. Another interesting direction is to apply the same kind of construction to other models than the six-vertex model. Of particular interest here would be the 'dilute version', obtained by staggering the Izergin–Korepin 19-vertex model.

In the CFT perspective, the expression for the toroidal partition function in terms of Coulombic partition functions generally leads to a classification of new minimal series of CFTs. It is possible, in principle, to follow this programme in the case of the $\mathbb{Z}_2$ model partition functions.

There are also some important questions about the physical interpretation of the $\mathbb{Z}_2$ Hamiltonian as a zig–zag spin chain. We have seen that non-Hermitian terms in the Hamiltonian play a role, but it could be that the model is in the same universality class as a well-defined, Hermitian spin-chain model. Additionally, at the Majumdar–Ghosh point, the gapped excitations above the ground state (spinons) could be studied more systematically, through a variational approach similar to [28].

In the context of the quadratic TL Hamiltonian (2.14), the $\mathbb{Z}_2$ model is an integrable point, governing the behaviour of a whole critical phase, as was shown numerically in [25]. Various features of this phase diagram still lead to open questions, such as the complete RG flow of (2.14) and the associated operators at the fixed points, but also the differences between the RSOS and loop formulations.

Acknowledgments

Appendix A. Physical quantities for holes

This appendix is about the analysis of the Bethe equations in the continuum limit. Here we prove equation (3.21), which gives the variation of a physical quantity A in the presence of a hole λ_h in the distribution $\{\lambda_j^{(0)}\}$. It is useful first to give the Fourier transform of the momentum and the kernels:

$$2\widehat{k'}(\omega) = \frac{2\pi \sinh(\pi/2 - \gamma)\omega}{\sinh(\pi\omega/2)}, \tag{A.1}$$

$$\widehat{K}^{(0)}(\omega) = -\frac{2\pi \sinh(\pi - 2\gamma)\omega}{\sinh \pi\omega}, \qquad \widehat{K}^{(\pm 1)}(\omega) = \frac{2\pi \sinh 2\gamma\omega}{\sinh \pi\omega}, \tag{A.2}$$

$$1 + \widehat{J}^{(+)}(\omega) = \frac{\sinh(\pi\omega/2)}{2\sinh(\pi/2 - \gamma)\omega \, \cosh \gamma\omega},$$

$$1 + \widehat{J}^{(-)}(\omega) = \frac{\cosh(\pi\omega/2)}{2\cosh(\pi/2 - \gamma)\omega \, \cosh \gamma\omega}. \tag{A.3}$$

The hole λ_h affects the Lieb equations (3.17):

$$\begin{cases} 2k' = (2\pi - K^{(0)}) \star \rho^{(0)} - K^{(1)} \star \rho^{(1)} + \frac{K^{(0)}(\lambda - \lambda_h)}{N}, \\[2mm] 2k' = -K^{(1)} \star \rho^{(0)} + (2\pi - K^{(0)}) \star \rho^{(1)} + \frac{K^{(1)}(\lambda - \lambda_h)}{N}, \end{cases} \tag{A.4}$$

Combining with the ground-state equation, we get

$$\rho^{(0)}(\lambda) + \rho^{(1)}(\lambda) - 2\rho_\infty(\lambda) = -\frac{1}{N} J^{(+)}(\lambda - \lambda_h). \tag{A.5}$$

The variation of A is given by

$$\begin{aligned} A - A_0 &= \int_{-\infty}^{\infty} d\lambda \, [\rho^{(0)}(\lambda) - \rho_\infty(\lambda)] \, \alpha(\lambda) \\ &\quad + \int_{-\infty}^{\infty} d\lambda \, [\rho^{(1)}(\lambda) - \rho_\infty(\lambda)] \, \alpha(\lambda) - \frac{1}{N}\alpha(\lambda_h) \\ &= \int_{-\infty}^{\infty} d\lambda (\rho^{(0)} + \rho^{(1)} - 2\rho_\infty)(\lambda) \, \alpha(\lambda) - \frac{1}{N}\alpha(\lambda_h) \\ &= -\frac{1}{N} \int_{-\infty}^{\infty} d\lambda \, J^{(+)}(\lambda - \lambda_h) \, \alpha(\lambda) - \frac{1}{N}\alpha(\lambda_h). \end{aligned} \tag{A.6}$$

Since $J^{(+)}$ is even, we get result (3.21).

Appendix B. Finite-size corrections

In this appendix, we introduce a variant of the Wiener–Hopf method [44], to calculate finite-size corrections to the energies from the analysis of the Bethe equations.

We consider combined magnetic excitations $(m^{(0)}, m^{(1)})$. Since the Bethe integer distributions $\{I_j^{(0,1)}\}$ are symmetric around zero, the bounds of the integrals in equation (3.17) are such that $C^{(0)} = B^{(0)}, C^{(1)} = B^{(1)}$. We can write

$$\begin{cases} 2k'(\lambda) = 2\pi \rho^{(a)}(\lambda) - \sum_{b=0,1} \int_{-B^{(b)}}^{+B^{(b)}} d\mu \, \rho^{(b)}(\mu) K^{(a-b)}(\lambda - \mu) \\[2mm] 2k'(\lambda) = 2\pi \rho_\infty(\lambda) - \int_{-\infty}^{\infty} d\mu \, \rho_\infty(\mu)(K^{(0)} + K^{(1)})(\lambda - \mu). \end{cases} \tag{B.1}$$

Applying the convolution by $(2\pi)^{-1}(\delta + J^{(\pm)})$ to symmetric and antisymmetric combinations, we get

$$\begin{cases} (\rho^{(0)} + \rho^{(1)} - 2\rho_\infty)(\lambda) = -\sum_{a=0,1} \int_{|\mu|>B^{(a)}} d\mu \, \rho^{(a)}(\mu) J^{(+)}(\lambda - \mu) \\[2mm] (\rho^{(0)} - \rho^{(1)})(\lambda) = -\sum_{a=0,1} (-1)^a \int_{|\mu|>B^{(a)}} d\mu \, \rho^{(a)}(\mu) J^{(-)}(\lambda - \mu). \end{cases}$$

Combining again the equations, we get

$$(\rho^{(a)} - \rho_\infty)(\lambda) = -\sum_{b=0,1} \int_{|\mu|>B^{(b)}} d\mu \, \rho^{(b)}(\mu) J^{(a-b)}(\lambda - \mu), \tag{B.2}$$

where $J^{(0)}, J^{(\pm 1)}$ are defined in (3.27). Let us define the symmetric/antisymmetric physical quantities

$$A^{(\pm)} \equiv \int_{-B^{(0)}}^{+B^{(0)}} d\lambda \, \rho^{(0)}(\lambda)\alpha(\lambda) \pm \int_{-B^{(1)}}^{+B^{(1)}} d\lambda \, \rho^{(1)}(\lambda)\alpha(\lambda). \tag{B.3}$$

The variation of $A^{(\pm)}$ with respect to the ground-state value $(A^{(\pm)})_\infty$ can be expressed as

$$A^{(+)} - A_\infty^{(+)} = -\sum_{a=0,1} \int_{|\mu|>B^{(a)}} \rho^{(a)} \times [(\delta + J^{(+)}) \star \alpha]\, d\mu$$

$$A^{(-)} - A_\infty^{(-)} = -\sum_{a=0,1} (-1)^a \int_{|\mu|>B^{(a)}} \rho^{(a)} \times [(\delta + J^{(-)}) \star \alpha]\, d\mu,$$

where we use the fact that $J^{(+)}$ and $J^{(-)}$ are even. Setting $\alpha(\lambda) = -1$ or $\alpha(\lambda) = \epsilon(\lambda)$, we get the charges and the energy:

$$\frac{m}{N} = [1 + \widehat{J}^{(+)}(0)] \sum_{a=0,1} \int_{|\mu|>B^{(a)}} d\mu\, \rho^{(a)}(\mu), \tag{B.4}$$

$$\frac{\widetilde{m}}{N} = [1 + \widehat{J}^{(-)}(0)] \sum_{a=0,1} (-1)^a \int_{|\mu|>B^{(a)}} d\mu\, \rho^{(a)}(\mu), \tag{B.5}$$

$$\frac{E - E_{\mathrm{gs}}}{N} = \sum_{a=0,1} \int_{|\mu|>B^{(a)}} d\mu\, \rho^{(a)}(\mu)\epsilon_d(\mu). \tag{B.6}$$

To solve the Lieb equations (B.2), we define the shifted densities: $g^{(a)}(\lambda) = \rho^{(a)}(B^{(a)}+\lambda)$ for $a = 0, 1$. Neglecting the terms from $\mu < 0$ (see [44]), we get the coupled Wiener–Hopf equations

$$g^{(a)}(\lambda) + \int_0^\infty d\mu\, g^{(b)}(\mu) J^{(a-b)}(\lambda - \mu + B^{(a,b)}) = \rho_\infty(B^{(a)} + \lambda),$$

where $B^{(a,b)} = B^{(a)} - B^{(b)}$. After Fourier transform

$$\begin{cases} (1 + \widehat{J}^{(0)})\, g_+^{(0)} + e^{-i\omega b}\widehat{J}^{(-1)}\, g_+^{(1)} + g_-^{(0)} = e^{-i\omega B^{(0)}}\widehat{\rho}_\infty \\ e^{i\omega b}\widehat{J}^{(1)}\, g_+^{(0)} + (1 + \widehat{J}^{(0)})\, g_+^{(1)} + g_-^{(1)} = e^{-i\omega B^{(1)}}\widehat{\rho}_\infty, \end{cases} \tag{B.7}$$

where $b = B^{(0)} - B^{(1)}$. We use the factorizations

$$1 + \widehat{J}^{(+)}(\omega) = \frac{1}{G_+(\omega)G_-(\omega)}, \qquad 1 + \widehat{J}^{(-)}(\omega) = \frac{1}{H_+(\omega)H_-(\omega)}, \tag{B.8}$$

where

$$G_+(\omega) = \sqrt{\frac{2\pi^2}{\pi - 2\gamma}}\, \frac{\Gamma(i\omega/2)}{\Gamma[(1/2 - \gamma/\pi)i\omega]\, \Gamma[1/2 + (\gamma/\pi)i\omega]}, \tag{B.9}$$

$$G_-(\omega) = G_+(-\omega), \tag{B.10}$$

$$H_+(\omega) = \frac{\sqrt{2\pi}\, \Gamma(1/2 + i\omega/2)}{\Gamma[1/2 + (1/2 - \gamma/\pi)i\omega]\, \Gamma[1/2 + (\gamma/\pi)i\omega]}, \tag{B.11}$$

$$H_-(\omega) = H_+(-\omega), \tag{B.12}$$

and we factorize the 2×2 matrix:

$$\mathbf{1} + \widehat{\mathbf{J}} \equiv \begin{pmatrix} 1 + \widehat{J}^{(0)} & e^{-i\omega b}\widehat{J}^{(-1)} \\ e^{i\omega b}\widehat{J}^{(1)} & 1 + \widehat{J}^{(0)} \end{pmatrix} = \mathbf{G}_-^{-1}\, \mathbf{G}_+^{-1}. \tag{B.13}$$

The matrices $\mathbf{G}_\pm$ read

$$\mathbf{G}_\pm = \frac{1}{2}\begin{pmatrix} G_\pm + H_\pm & \mathrm{e}^{-\mathrm{i}\omega b}(G_\pm - H_\pm) \\ \mathrm{e}^{\mathrm{i}\omega b}(G_\pm - H_\pm) & G_\pm + H_\pm \end{pmatrix}, \tag{B.14}$$

$$\mathbf{G}_\pm^{-1} = \frac{1}{2}\begin{pmatrix} G_\pm^{-1} + H_\pm^{-1} & \mathrm{e}^{-\mathrm{i}\omega b}(G_\pm^{-1} - H_\pm^{-1}) \\ \mathrm{e}^{\mathrm{i}\omega b}(G_\pm^{-1} - H_\pm^{-1}) & G_\pm^{-1} + H_\pm^{-1} \end{pmatrix}. \tag{B.15}$$

We can write the system (B.7) as

$$(\mathbf{1} + \widehat{\mathbf{J}})\begin{pmatrix} g_+^{(0)} \\ g_+^{(1)} \end{pmatrix} + \begin{pmatrix} g_-^{(0)} \\ g_-^{(1)} \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\omega B^{(0)}}\widehat{\rho}_\infty\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\omega b} \end{pmatrix}. \tag{B.16}$$

We multiply by $\mathbf{G}_-$:

$$\mathbf{G}_+^{-1}\begin{pmatrix} g_+^{(0)} \\ g_+^{(1)} \end{pmatrix} + \mathbf{G}_-\begin{pmatrix} g_-^{(0)} \\ g_-^{(1)} \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\omega B^{(0)}}\widehat{\rho}_\infty\,\mathbf{G}_-\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\omega b} \end{pmatrix}. \tag{B.17}$$

The solution is given in terms of the pole $\omega_0 = -\mathrm{i}\pi/(2\gamma)$ for $\widehat{\rho}_\infty$ and the residue $r_0 = \mathrm{Res}(\widehat{\rho}_\infty, \omega_0)$:

$$\begin{pmatrix} g_+^{(0)} \\ g_+^{(1)} \end{pmatrix} = \mathbf{G}_+\left[\mathrm{e}^{-\mathrm{i}\omega B^{(0)}}\widehat{\rho}_\infty\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\omega b} \end{pmatrix}\right]_+$$

$$= -\frac{r_0\zeta^{(0)}}{\omega_0 - \omega}\mathbf{G}_+(\omega)\mathbf{G}_-(\omega_0)\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\omega_0 b} \end{pmatrix}$$

$$= -\frac{r_0\zeta^{(0)}G_-(\omega_0)}{\omega_0 - \omega}\mathbf{G}_+(\omega)\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\omega_0 b} \end{pmatrix}, \tag{B.18}$$

where $\zeta^{(a)} = \mathrm{e}^{-\mathrm{i}\omega_0 B^{(a)}}$. So the magnetic charges are given by

$$\frac{m}{N} = -\frac{2r_0 G_-(\omega_0)}{\omega_0 G_-(0)}(\zeta^{(0)} + \zeta^{(1)}), \tag{B.19}$$

$$\frac{\widetilde{m}}{N} = -\frac{2r_0 G_-(\omega_0)}{\omega_0 H_-(0)}(\zeta^{(0)} - \zeta^{(1)}). \tag{B.20}$$

The total energy is

$$\frac{E - E_{\mathrm{gs}}}{N} = 2\mathrm{i}\,\mathrm{Res}(\widehat{\epsilon}_d, \omega_0)\,[\zeta^{(0)}g_+^{(0)}(-\omega_0) + \zeta^{(1)}g_+^{(1)}(-\omega_0)]$$

$$= 2\mathrm{i}\pi\sin 2\gamma\,\frac{r_0^2 G_-(\omega_0)G_+(-\omega_0)}{\omega_0}[(\zeta^{(0)})^2 + (\zeta^{(1)})^2]$$

$$= \frac{2\pi v}{8}\left[\left(\frac{G_-(0)m}{N}\right)^2 + \left(\frac{H_-(0)\widetilde{m}}{N}\right)^2\right]$$

$$= \frac{2\pi v}{8}\left\{\frac{1}{1 + \widehat{J}^{(+)}(0)}\left(\frac{m}{N}\right)^2\frac{1}{1 + \widehat{J}^{(-)}(0)}^{-1}\left(\frac{\widetilde{m}}{N}\right)^2\right\}. \tag{B.21}$$

Using expressions (A.3) for $\widehat{J}^{(\pm)}$, we get the critical exponents given in (3.31). A similar calculation with $C^{(a)} \neq B^{(a)}$ would give the electric critical exponents.

Appendix C. Bosonic partition functions and Jacobi's theta functions

C.1. Free boson on a torus

Let us recall some known results on the free boson theory on a torus [45]. We denote by τ the modular ratio of the torus, and we write $q = e^{2i\pi\tau}$. The free boson is defined by the action $\mathcal{A}$ and the partition function Z_0:

$$\mathcal{A}[\varphi] = \frac{g}{4\pi} \int d^2x \, |\nabla\varphi|^2, \tag{C.1}$$

$$Z_0(g) = \int [D\varphi] \, \exp(-\mathcal{A}[\varphi]) = \sqrt{\frac{g}{\operatorname{Im}\tau}} \frac{1}{|\eta(\tau)|^2}, \tag{C.2}$$

where g is the coupling constant, and $\eta(\tau)$ is the Dedekind function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{C.3}$$

When defects $\delta\varphi, \delta'\varphi$ are introduced on the boundaries, this defines the partition function $Z_{m,m'}$, with m, m' integers:

$$Z_{m,m'}(g) = \int_{\substack{\delta\varphi=2\pi m \\ \delta'\varphi=2\pi m'}} [D\varphi] \, \exp(-\mathcal{A}[\varphi])$$

$$= Z_0(g) \, \exp\left(-\frac{\pi g|m' - m\tau|^2}{\operatorname{Im}\tau}\right). \tag{C.4}$$

A Poisson summation of (C.4) yields

$$\sum_{m'\in\mathbb{Z}} e^{i\alpha m'} Z_{m,m'}(g)$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{k\in\mathbb{Z}+\alpha/(2\pi)} q^{(k/\sqrt{g}+m\sqrt{g})^2/4} \, \bar{q}^{(k/\sqrt{g}-m\sqrt{g})^2/4}. \tag{C.5}$$

C.2. Jacobi theta functions

The Jacobi theta functions are defined as

$$\theta_1(\tau) = -i\sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2/2}, \qquad \theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2},$$

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \qquad\qquad \theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}. \tag{C.6}$$

They obey the algebraic relations:

$$\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) = 2\eta(\tau)^3, \tag{C.7}$$

$$\sqrt{\theta_3(\tau)\theta_4(\tau)} = \theta_4(2\tau). \tag{C.8}$$

We denote the Jacobi partition functions by

$$Z_\nu = \left|\frac{\theta_\nu(\tau)}{\eta(\tau)}\right|, \qquad \nu = 2, 3, 4. \tag{C.9}$$

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The Ising partition functions $\mathcal{Z}_{r,r'}$ are related to the Jacobi ones by

$$
\begin{aligned}
Z_2 &= \mathcal{Z}_{1,0} + \mathcal{Z}_{1,1} \\
Z_3 &= \mathcal{Z}_{0,1} + \mathcal{Z}_{1,0} \\
Z_4 &= \mathcal{Z}_{0,1} + \mathcal{Z}_{1,1} \\
\mathcal{Z}_{0,0} &= \mathcal{Z}_{0,1} + \mathcal{Z}_{1,0} + \mathcal{Z}_{1,1}.
\end{aligned}
\tag{C.10}
$$

Appendix D. Partition functions for the staggered models

D.1. Twisted vertex model and Potts model

Starting from the untwisted partition function $Z(g)$, we can proceed as in [45], to construct the twisted partition function and the Potts partition function. The partition function where non-contractible loops have a weight $\widehat{n} = 2\cos\phi$ is given by

$$
\widehat{Z}(g, \phi) = 2\left(A \sum_{m,m' \text{ even}} + B \sum_{m \text{ even}, m' \text{ odd}} \right.
$$
$$
\left. + C \sum_{m \text{ odd}, m' \text{ even}} + D \sum_{m,m' \text{ odd}} \right) Z_{m,m'}(g) \, \cos(2\phi \, m \wedge m'),
\tag{D.1}
$$

where $m \wedge m'$ denotes the greatest common factor between m and m'. In particular, for $\phi = \pi/2$ and $\phi = \pi/4$, we have

$$
\widehat{Z}(g, \pi/2) = 2\left(A \sum_{m,m' \text{ even}} - B \sum_{m \text{ even}, m' \text{ odd}} \right.
$$
$$
\left. - C \sum_{m \text{ odd}, m' \text{ even}} - D \sum_{m,m' \text{ odd}} \right) Z_{m,m'}(g),
\tag{D.2}
$$

$$
\widehat{Z}(g, \pi/4) = A\left[\sum_{m,m' \in \mathbb{Z}} Z_{m,m'}\left(\frac{1}{16g} \right) - 2 \sum_{m,m' \in 2\mathbb{Z}} Z_{m,m'}(g) \right].
\tag{D.3}
$$

The Q-state Potts partition function has an extra term due to clusters with cross geometry [45]:

$$
Z_{\text{Potts}}(Q) = \widehat{Z}(g, \pi e_0) + \tfrac{1}{2}(Q-1)\widehat{Z}(g, \pi/2),
\tag{D.4}
$$

where

$$
\sqrt{Q} = 2\cos\gamma, \qquad 0 < \gamma < \frac{\pi}{2}, \qquad g = \frac{\pi - 2\gamma}{2\pi}, \qquad e_0 = \frac{\gamma}{\pi}.
\tag{D.5}
$$

D.2. Particular values of Q

- The case $Q = 2$

  This provides a good check of result (4.14), since the Potts model arising from the staggered vertex model is equivalent, on the lattice, to the usual critical Ising model. Using (D.4),

$$
Z_{\text{Potts}}(Q = 2) = \left[(A - B) \sum_{m \text{ even}, m' \text{ odd}} + (A - C) \sum_{m \text{ odd}, m' \text{ even}} \right.
$$
$$
\left. + (A - D) \sum_{m,m' \text{ odd}} \right] Z_{m,m'}(1/4).
\tag{D.6}
$$

Now the sums on m, m' can be expressed in terms of the Z_ν:

$$\sum_{m \text{ even}, m' \text{ odd}} Z_{m,m'}(1/4) = \frac{1}{2} Z_3 Z_4$$

$$\sum_{m \text{ odd}, m' \text{ even}} Z_{m,m'}(1/4) = \frac{1}{2} Z_2 Z_3 \tag{D.7}$$

$$\sum_{m,m' \text{ odd}} Z_{m,m'}(1/4) = \frac{1}{2} Z_2 Z_4.$$

We obtained the first identity by using (C.7), and the two others by expanding the square of the left-hand sides. Combining (D.7) with (4.8) and (C.8), we get

$$Z_{\text{Potts}}(Q = 2) = \tfrac{1}{2}(Z_2 + Z_3 + Z_4) = Z_{\text{Ising}}, \tag{D.8}$$

so we correctly find the Ising partition function.

- The case $Q = 1$

  This case is *a priori* a bit intriguing. The partition function of the Potts model is then a trivial object (since there is only one state available for the whole lattice), while the general formulae for the central charge give in this particular case $c = -2$ ($Q = 1$ so $\gamma = \pi/3$, $g = 1/6$, $e_0 = 1/3$). This discrepancy occurs for the same reason as in the Berker–Kadanoff phase [4]: the level of the transfer matrix corresponding to a trivial partition function (and hence, formally, $c = 0$) is very high in the spectrum, while the level generically dominating the thermodynamics (but which disappears *right at $Q = 1$* by quantum group truncation) corresponds to $c = -2$ (this means that the free energy is a discontinuous function of Q or of the boundary conditions [4]).

  Let us now see the mechanism in more detail. The ground-state energy of our system in the untwisted case is twice the ground-state energy of the antiferromagnetic *XXZ* model with $\Delta_0 = -\cos 2\gamma$. In the case $Q = 1$ we have $\Delta_0 = \frac{1}{2}$. The antiferromagnetic *XXZ* model with this value of the anisotropy is related to the Potts model at $Q = 1$ on the 'non-physical self-dual line' [33]. Recall that, meanwhile, the Potts model on the usual self-dual line is related to the antiferromagnetic *XXZ* chain at $2\Delta = -\sqrt{Q}$, so $\Delta = -\frac{1}{2}$ in the case $Q = 1$.

  Now we know that the energies of the antiferromagnetic *XXZ* at $\Delta = -\frac{1}{2}$ are *minus* the energies of the antiferromagnetic *XXZ* at $\Delta = \frac{1}{2}$ (this is the general mapping between H_Δ and $-H_{-\Delta}$). The ground-state energy of the antiferromagnetic *XXZ* at $\Delta = \frac{1}{2}$ is the same, per unit length in the thermodynamic limit, as the one of the twisted antiferromagnetic *XXZ*, i.e. the ground-state energy of the percolation problem, i.e. $E_0 = 0$ in the proper normalization. We thus conclude that the eigenvalue 'corresponding to $Z = 1$' in our spectrum is the *most excited among the subset of symmetric states*.

  It is useful to see this mechanism at the level of partition functions as well. Start from (D.1) and set $Q = 1$, $\phi = \pi/3$. Then there is no contribution from the cross-geometry clusters. Since $\cos 2\pi/3 = \cos 4\pi/3 = -\frac{1}{2}$, we have

$$\sum_{m,m'} Z_{m,m'}(g) \cos(2\phi\, m \wedge m') = \left(\frac{3}{2} \sum_{m,m' \in 3\mathbb{Z}} - \frac{1}{2} \sum_{m,m'} \right) Z_{m,m'}.$$

Moreover, $m = 3p$ is odd (resp. even) iff p is odd (resp. even). Finally, $Z_{3p,3p'}(g) = Z_{p,p'}(9g)/3$. So we can rewrite

$$Z_{\text{Potts}}(Q = 1) = \left(A \sum_{m,m' \text{ even}} + B \sum_{m \text{ even},m' \text{ odd}} \right.$$

$$\left. + C \sum_{m \text{ odd},m' \text{ even}} + D \sum_{m,m' \text{ odd}} \right) [Z_{m,m'}(9g) - Z_{m,m'}(g)]. \qquad (D.9)$$

We can recombine terms using expressions for the A, B, C, D in terms of the Z_ν:

$$Z_{\text{Potts}}(Q = 1) = \frac{1}{2}(A - B - C - D)$$

$$+ (B - D) \sum_{m \text{ even},m'} [Z_{m,m'}(3/2) - Z_{m,m'}(1/6)]$$

$$+ (C - D) \sum_{m,m' \text{ even}} [Z_{m,m'}(3/2) - Z_{m,m'}(1/6)],$$

where we have specialized to $g = 1/6$ and used Euler's identity:

$$\sum_{m,m' \text{even}} [Z_{m,m'}(3/2) - Z_{m,m'}(1/6)] = \frac{1}{2} [Z_c(6) - Z_c((2/3)]$$

$$= 1. \qquad (D.10)$$

Both of the terms

$$\sum_{m \text{ even},m'} [Z_{m,m'}(3/2) - Z_{m,m'}(1/6)] \qquad (D.11)$$

and

$$\sum_{m,m' \text{ even}} [Z_{m,m'}(3/2) - Z_{m,m'}(1/6)] \qquad (D.12)$$

can be shown to vanish exactly. We conclude that

$$Z_{\text{Potts}}(Q = 1) = 0. \qquad (D.13)$$

This means there are exact cancellations among states in the low-energy spectrum: the (unique) state that would correspond to the trivial partition function is very highly excited and does not contribute to the conformal partition function (at $c = -2$ in this case).

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